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The Log-Rank Conjecture for Read-*k* XOR Functions

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The log-rank conjecture states that the deterministic communication complexity of a Boolean function g (denoted by $D^{cc}(g)$) is polynomially related to the logarithm of the rank of the communication matrix M_g where M_g is the communication matrix defined by $M_g(x, y)=g(x, y)$. An XOR function $G:\{0, 1\}^n\times\{0, 1\}^n\rightarrow\{0, 1\}$ with respect to $g:\{0, 1\}^n\rightarrow\{0, 1\}$ is a function defined by $G(x, y)= g(x\oplus y)$. It is well-known that $||\hat{g}||_0 = \mathbf{rank}(M_G)$ where $||\hat{g}||_0$ is the Fourier 0-norm of g, M_G is the communication matrix defined by $M_G(x, y)=G(x, y)$, and $\mathbf{rank}(M_G)$ is the dimension of the row space of M_G over reals. The log-rank conjecture for XOR functions is equivalent to the question whether the deterministic communication complexity of an XOR function G with respect to a function G is polynomially related to the logarithm of the Fourier sparsity of g, namely $D^{cc}(G)=\log^c(||\hat{g}||_0)$ for a fixed constant c. Previously, the log-rank conjecture holds for XOR functions with respect to symmetric functions, linear threshold functions, monotone functions, AC^0 functions, and constant-degree polynomials over F_2.

In this paper, we consider a special class of functions called read-k polynomials over F_2. We study the communication complexity of the XOR function G with respect to a read-k polynomial g. We show that $D^{cc}(G)=O(kd^2\log(||\hat{g}||_1))$ where d is the F_2-degree of g. By the well-known bound that $d \leq \log(||\hat{g}||_0)$, we conclude that $D^{cc}(G) = O(k\log^3(||\hat{g}||_0))$. In particular, if $k=\log^{O(1)}(||\hat{g}||_0)$, then we have $D^{cc}(G)=O(\log^{O(1)}(\mathbf{rank}(M_G)))$.

***Keywords*:** communication complexity, the log-rank conjecture, read-*k* polynomials, read-*k* XOR functions, fourier sparsity

1. INTRODUCTION

One of the main research topics in theoretical computer science is to study the communication complexity of Boolean functions. In his seminal work [18], Yao introduced the two-party communication model for computing Boolean functions. In this model, two parties Alice and Bob cooperate to compute a function $g:X\times Y\rightarrow\{0, 1\}$. Alice has an input $x\in X$, Bob has an input $y\in Y$, and they compute the output $g(x, y)$ by exchanging messages. The least number of message bits they exchange on the worst-case input is the communication complexity of the function g. If the protocol is deterministic, then we call it deterministic communication complexity of g, denoted by $D^{cc}(g)$. The lower bound of $D^{cc}(g)$ is studied by Mehlhorn and Schmidt first. In [14], they showed that $D^{cc}(g) \geq \log(\mathbf{rank}(M_g))$ where M_g is the communication matrix defined by $M_g(x, y)=g(x, y)$ and $\mathbf{rank}(M_g)$ is the dimension of the row space of M_g over reals. In 1988, Lovász and Saks [8] proposed the log-rank conjecture which states that there exists a fixed constant c such that $D^{cc}(g) = \log^c(\mathbf{rank}(M_g))$ for any Boolean function g. The log-rank conjecture provides a way to study the deterministic communication complexity of Boolean functions by calculating the rank of the com-

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munication matrix. Despite its great importance, it is hard to prove the log-rank conjecture. In [4, 5], it is shown that $D^{cc}(g) \leq \log(4/3)\mathbf{rank}(M_g)$. Recently, in [3], Ben-Sasson, Lovett, and Ron-Zewi gave a conditional result which shows that $D^{cc}(g) = O(\mathbf{rank}(M_g)/\log(\mathbf{rank}(M_g)))$ by assuming the Polynomial Freiman-Ruzsa conjecture. In addition, Lovett unconditionally shows that $D^{cc}(g) = O(\mathbf{rank}(M_g))^{1/2}\log(\mathbf{rank}(M_g)))$ [11].

To attack the log-rank conjecture, one may consider some special classes of functions. In [21], Zhang and Shi studied a special class of Boolean functions called XOR functions defined as follows.

Definition 1: A function $G:\{0, 1\}^n\times\{0, 1\}^n\rightarrow\{0, 1\}$ is called an XOR function with respect to a function $g:\{0, 1\}^n\rightarrow\{0, 1\}$ if $G(x, y) = g(x\oplus y)$ for all $x, y\in\{0, 1\}^n$. We denote G by $g \circ \oplus$.

There are many interesting XOR functions such as Hamming Distance and Equality. For an XOR function G with respect to g, an important observation is that the rank of the communication matrix M_G is identical to the number of non-zero Fourier coefficients of g. This number is called the Fourier sparsity of g. and denoted by $||\hat{g}||_0$. Now, for any XOR function $G = g \circ \oplus$, the log-rank conjecture is equivalent to the question whether $D^{cc}(g) = \log^c(||\hat{g}||_0)$ for a fixed constant c. Based on this observation, the log-rank conjecture for XOR functions draws much attention recently [6, 7, 9, 10, 13, 15-17, 19-21]. Among these works, some nice results are obtained and listed below. In [20], Zhang and Shi showed that the log-rank conjecture holds for symmetric XOR functions. In [13], Montanaro and Osborne showed that the log-rank conjecture holds for monotone XOR functions and linear threshold XOR functions. In [6], Kulkarni and Santha showed that the log-rank conjecture holds for AC^0 XOR functions. More recently, in [17], Tsang *et al.* showed that the log-rank conjecture holds for XOR functions with respect to constant-degree polynomials over F_2. In these works, many proof approaches are proposed. One of them is the approach based on parity decision trees which is initiated from [21]. We describe this approach in the next subsection.

1.1 The Approach Based on Parity Decision Trees

In [13, 21], the authors proposed a good way to study the log-rank conjecture for XOR functions $g \circ \oplus$. In this approach, an efficient parity decision tree (PDT) is designed to compute G and then is converted into an efficient two-party communication protocol. The notion of parity decision trees is a generalized notion of traditional decision trees. In each internal node, a parity decision tree allows one to query the parity of any subset of input variables while a decision tree only allows one to query just one input variable. The deterministic parity decision tree complexity of g, denoted by $D_\oplus(g)$, is the least number of queries required on a worst-case input by a parity decision tree that computes g. It is known that $D^{cc}(g \circ \oplus) \leq 2\, D_\oplus(g)$. Therefore, in order to prove the log-rank conjecture for XOR functions, it is sufficient to show that $D_\oplus(g) \leq \log^{O(1)}(||\hat{g}||_0)$ for any Boolean function g. Although parity decision trees provide a nice way to prove the log-rank conjecture, designing an efficient PDT algorithm for a given function is still challenging.

One way to upper bound $D_\oplus(g)$ is to use the decision tree complexity of g, denoted by $D(g)$. A well-known fact [2] shows that $D(g)=O((\deg(g))^4)$ where $\deg(g)$ is the degree of the Fourier polynomial (over reals) of g, that is $\deg(g) = \max_{\alpha:\hat{g}(\alpha)\neq 0}|\alpha|$. Moreover, Midrijanis showed that $D(g) = O((\deg(g))^3)$ [12]. From that, one can obtain $D_{\oplus}(g) \leq D(g) \leq \log^{O(1)}(||\hat{g}||_0)$ if $\deg(g) = \log^{O(1)}(||\hat{g}||_0)$. However, some sparse polynomials may have high degree.

Another approach is based on the degree of the polynomial over F_2. Note that every Boolean function G can be written as a polynomial over F_2. Let $\deg_2(g)$ (called F_2-degree of g) denote the minimum degree over these polynomials for computing g. Given any Boolean function g, it is shown in [1] that the F_2-degree of g is at most the logarithm of its Fourier sparsity, namely $\deg_2(g) \leq \log_2(||\hat{g}||_0)$. Based on this observation, Tsang *et al.* [17] defined the notion of the polynomial rank of Boolean functions and used it to design an efficient PDT algorithm. The polynomial rank of g is the minimum number r such that g can be expressed as $g = \ell_1 g_1 \oplus \ldots \oplus \ell_r g_r \oplus g_0$ where $\deg_2(\ell_i) = 1$ and $\deg_2(g_i) < \deg_2(g)$ for any i. We denote this minimum integer r as $\mathbf{rank}(g)$.

Tsang *et al.* [17] showed that, for any Boolean function g with F_2-degree d, $\mathbf{rank}(g) = O(2^{d^2/2}\log^{d-2}(||\hat{g}||_1))$. From this, they showed that, for any XOR function G with respect to a polynomial g with F_2-degree d, $D^{cc}(G) = O(2^{d^2/2}\log^{d-2}(||\hat{g}||_1))$.

As a result, if G is an XOR function with respect to a constant-F_2-degree polynomial g, then $D^{cc}(G) = \log^{O(1)}(\mathbf{rank}(M_G))$. Note that the hidden constant in the exponent of logarithm depends on the F_2-degree of g.

1.2 Our Results

In this paper, we consider read-k polynomials over F_2. Let $x_1, \ldots, x_n$ be Boolean variables. A polynomial over F_2 is an Exclusive-Or ($\oplus$) of AND gates on n variables $x_1, \ldots, x_n$. The F_2-degree of a polynomial is defined as the maximum fanin of its AND gates. The F_2-degree of a function g is the minimum degree over these polynomials for computing g. A polynomial is called a read-k polynomial if each variable appears in at most k AND gates. We show that, if g is a read-k polynomial with $\deg_2(g)=d$, then $D^{cc}(G) = O(d^2k\log(||\hat{g}||_1))$ where G is the XOR function with respect to g. In particular, if $k \leq \log^c(||\hat{g}||_1)$ for a fixed constant c, then we conclude that $D^{cc}(G)=O(d^2\log^{c+1}(||\hat{g}||_1))$. Therefore, for an XOR function G with respect to a read-k polynomial g with degree d and with $k \leq \log^c(||\hat{g}||_1)$, our bound on $D^{cc}(G)$ is better than the bound of Tsang *et al.* [17]. In addition, if $k \leq \log^c(||\hat{g}||_0)$, then we conclude that $D^{cc}(G)=\log^{O(1)}(||\hat{g}||_0)$. Hence the log-rank conjecture holds for XOR functions with respect to read-k polynomials with $k \leq \log^{O(1)}(||\hat{g}||_0)$.

1.3 Organization of This Paper

In Section 2, we define read-k polynomials and derive some of its properties. Moreover, some necessary definitions are given there. In Section 3, we prove that the log-rank conjecture holds for XOR functions with respect to read-k polynomials for small k.

2. PRELIMINARIES

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For a function $g:\{0, 1\}^n \rightarrow \{0, 1\}$, let $g^{\pm}:\{0,$

$1\}^n \rightarrow \{-1, 1\}$ be the function defined by $g^{\pm}=1-2g$. Each Boolean function $g:\{0, 1\}^n \rightarrow \{0, 1\}$ can be expressed as a polynomial over F_2 and let $\deg_2(g)$ denote the minimum degree over F_2-polynomials computing g. Let e_i be the n-bit vector whose ith entry is 1 and the other entries are all 0. For a Boolean function $g:\{0, 1\}^n \rightarrow \{0, 1\}$ and a vector $t \in \{0, 1\}^n \backslash \{0^n\}$, we define the derivative of g with respect to t by $\Delta_t g(x) = g(x \oplus t) \oplus g(x)$. An affine subspace is a set $H \subseteq \{0, 1\}^n$ such that $H = a \oplus V$ for a vector subspace $V \subseteq \{0, 1\}^n$ and an n-bit string a. The dimension of an affine subspace H associated with a vector subspace V is defined by $\dim(H) \doteq \dim(V)$ and the co-dimension of H, denoted by $\mathrm{codim}(H)$, is defined by $\mathrm{codim}(H) \doteq n - \dim(H)$.

2.1 Read-*k* Polynomials Over F_2

In this paper, we consider read-k polynomials over F_2. A polynomial $g(x_1, \ldots, x_n)$ over F_2 is an Exclusive-Or ($\oplus$) of AND gates on n variables $x_1, \ldots, x_n$. A polynomial is called a read-k polynomial if each variable x_i appears in at most k AND gates. Given an input $x \in \{0, 1\}^n$, we define $x^{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. The following two lemmas will be useful in the next section.

Lemma 1: Let $g:\{0, 1\}^n \rightarrow \{0, 1\}$, g_0, g_1, g_2: $\{0, 1\}^{n-1} \rightarrow \{0, 1\}$ be Boolean functions such that $g(x) = x_i g_0(x^{-i}) \oplus \neg(x_i) g_1(x^{-i}) \oplus g_2(x^{-i})$. Then $\Delta_{e_i} g(x) = g_0(x^{-i}) \oplus g_1(x^{-i})$.

***Proof*:** The differentiation of g with respect to e_i is as follows.

$$\begin{aligned}
\Delta_{e_i} g(x) &= g(e_i \oplus x) \oplus g(x) \\
&= (1 \oplus x_i) g_0(x^{-i}) \oplus \neg(1 \oplus x_i) g_1(x^{-i}) \oplus g_2(x^{-i}) \oplus g(x) \\
&= g_0(x^{-i}) \oplus g_1(x^{-i}) \oplus x_i g_0(x^{-i}) \oplus (1 \oplus x_i) g_1(x^{-i}) \oplus g_2(x^{-i}) \oplus g(x) \\
&= g_0(x^{-i}) \oplus g_1(x^{-i}) \oplus g(x) \oplus g(x) \\
&= g_0(x^{-i}) \oplus g_1(x^{-i}).
\end{aligned}$$

❑

The following lemma is straightforward.

Lemma 2: Given a Boolean function $g:\{0, 1\}^n \rightarrow \{0, 1\}$, if $\Delta_{e_i} g(x)$ is constant for any i, then g is a linear function.

2.2 Fourier Analysis

For any binary vector $s \in \{0, 1\}^n$, define $\chi_s(x) = (-1)^{s \cdot x}$. Given any function $g:\{0, 1\}^n \rightarrow \mathbb{R}$, the Fourier coefficients of g are defined by $\hat{g}(s) = 2^{-n} \Sigma_x f(x) \chi_s(x)$. By these coefficients, each function $g:\{0, 1\}^n \rightarrow \mathbb{R}$ can be written as $g(x) = \Sigma_s \hat{g}(s) \chi_s(x)$. For any $p > 0$, we define the L_p norm of g by $\|\hat{g}\|_p = (\Sigma_s |\hat{g}(s)|^p))^{1/p}$. In the case that $p=0$, we call $\|\hat{g}\|_0$ the Fourier sparsity of g. For any Boolean function g, its Fourier L_1 norm is less than or equal to the square root of its Fourier L_0 norm. We have the following useful facts and lemma.

Fact 1: For any Boolean function g, $\|\hat{g}\|_0 \leq (\|\hat{g}\|_0)^{1/2}$.
Fact 2: Let $g^{\pm} = 1-2g$. Then we have $\hat{g^{\pm}}(s) = -2\hat{g}(s)$ if $s \neq 0^n$ and $1-2\hat{g}(0^n)$ if $s = 0^n$.
Fact 3: Let g be a Boolean function. If $\|\hat{g^{\pm}}\|_1 \leq 1$, then g is a linear function.

Note that each Boolean function $g:\{0, 1\}^n\rightarrow\{0, 1\}$ can be expressed as a polynomial over F_2. One can bound $\deg_2(g)$ by logarithm of the Fourier zero norm of g by the following well-known fact.

Fact 4: ([1]) For any Boolean function g, $\deg_2(g) \leq \log(\|\hat{g}\|_0)$.

The following lemma is proved implicitly in [17]. We will use it later.

Lemma 3: ([17]) Suppose that, for any Boolean function $g:\{0, 1\}^n\rightarrow\{0, 1\}$, there exists a vector $c\in\{0, 1\}^n$ such that $\Delta_c g$ is non-constant. For any $b\in\{0, 1\}^n$, let H_b denote an affine subspace where $\Delta_c g|_{H_b}=b$ and $g_b \doteq g|_{H_b}$. Then $\min\{\|\hat{g}_0^{\pm}\|_1, \|\hat{g}_1^{\pm}\|_1\} \leq (\|\hat{g}^{\pm}\|_1)/2$.

2.3 Deterministic Communication Complexity

Given a function G: $\{0, 1\}^n\times\{0, 1\}^n\rightarrow\{0, 1\}$, two parties Alice who holds an input x and Bob who holds an input y cooperate to compute the output $G(x, y)$ by exchanging communication bits. The deterministic communication complexity of the function G, denoted by $D^{cc}(G)$, is the the least communication bits which are needed by Alice and Bob to compute the function G on any input (x, y). Let M_G denote the communication matrix defined by $M_G \doteq [G(x, y)]_{x,y}$. A well known result of Mehlhorn and Schmidt [14] shows that $\log(\mathbf{rank}(M_G)) \leq D^{cc}(G)$. In [8], Lovász and Saks proposed the log-rank conjecture which asserts that there is a fixed constant c such that $D^{cc}(G) \leq \log^c(\mathbf{rank}(M_G))$ for any Boolean function G. For an XOR function G with respect to a function g, the rank of the communication matrix M_G is equal to the Fourier L_0 norm of the function g.

Fact 5: ([20]) If G is an XOR function with respect to a function $g:\{0, 1\}^n\rightarrow\{0, 1\}$, then $\mathbf{rank}(M_G)= \|\hat{g}\|_0$.

2.4 Parity Decision Tree

A parity decision tree (PDT) for a function $g:\{0, 1\}^n\rightarrow\{0, 1\}$ is a binary tree where each internal node is associated with a linear function and each leaf is associated with an answer bit. The function value $g(x)$ is computed by a parity decision tree as follows. The computation starts from the root and follows a path down to a leaf. At each internal node, one queries the associated linear function and determines which branch to take according to the answer of the query. Once reaching a leaf, one outputs the associated answer bit as the function output value $g(x)$. The complexity of deterministic parity decision trees for computing $g(x)$, denoted by $D_{\oplus}(g)$, is the least height of PDTs that compute g. For XOR functions, the relationship between PDT and communication complexity is given in the following fact.

Fact 6: ([13]) If G is an XOR function with respect to a function g, then $D^{cc}(G)\leq 2D_{\oplus}(g)$.

Given a Boolean function g and an input x, the parity certificate complexity of g on x is defined by

$$C_{\oplus}(g, x) = \min\{\mathrm{codim}(H): H \text{ is an affine subspace with } x\in H \text{ and } g|_H \text{ is constant}\}.$$

The minimum parity certificate complexity $C_{\oplus,\min}(g)$ is defined by

$$C_{\oplus,\min}(g) \doteq \min_x C_{\oplus}(g, x).$$

The connection between $C_{\oplus,\min}(g)$ and $D_{\oplus}(g)$ is provided by Tsang *et al.* in the following lemma.

Lemma 4: ([17]) For any Boolean function $g:\{0, 1\}^n\to\{0, 1\}$ and any $b\in\{0, 1\}$, if there exist two non-negative constants c_1, c_2 such that $C_{\oplus,\min}(g)\leq(\deg_2(g))^{c_1}\log^{c_2}(\|\hat{g}\|_b)$, then $D_{\oplus}(g)\leq(\deg_2(g))^{1+c_1}\log^{c_2}(\|\hat{g}\|_b)$ and $D^{cc}(g\circ\oplus)\leq 2(\deg_2(g))^{1+c_1}\log^{c_2}(\|\hat{g}\|_b)$ where $\|\hat{g}\|_b$ denotes the Fourier L_b norm of g.

3. DETERMINISTIC COMMUNICATION COMPLEXITY OF XOR FUNCTIONS WITH RESPECT TO READ-K POLYNOMIALS FOR SMALL K

In this section, we show that the log-rank conjecture holds for every read-k polynomial g over F_2 with $k=\log^{O(1)}(\|\hat{g}\|_0)$.

Theorem 1: Suppose that g is a read-k polynomial and $\deg_2(g)=d$. Then $C_{\oplus,\min}(g)=O(dk \log((\|\hat{g}\|_1))$.

Proof: Given a subset $S\subseteq[n]$, let $T_S(x)\doteq\wedge_{i\in S}x_i$. We express $g(x)$ as $\oplus_{i\in[m]}T_{S_i}(x)$ for m subsets $S_1, S_2, \ldots, S_m\subseteq[n]$ where $|\{S_i : j\in S_i\}|\leq k$ for each $j\in[n]$. Since $\deg_2(g)=d$, we have $|S_i| \leq d$ for all $i\in[m]$. Suppose g is not a linear function. Otherwise, g can be set as a constant by only one linear restriction. So, by Lemma 2, there is a vector e_i such that $\Delta_{e_i}(g)$ is non-constant. Then, by Lemma 1 and the assumption that g is read-k, there exist $t\leq k$ subsets $S_1', \ldots, S_t'$ such that $i\notin S_j'$ and $S_j'\cup\{i\}\in\{S_1, \ldots, S_m\}$ for each $j\in[t]$, and

$$\Delta_{e_i}(g)(x) = \oplus_{i\in[m]}T_{S_i}(x).$$

Let $x_{i(1,1)}, \ldots, x_{i(1,p_1)}$ be Boolean variables appearing in the function $\Delta_{e_i}(g)(x)$ where $p_1\leq t(d-1)\leq k(d-1)$. For any $b\in\{0, 1\}$, we can set these Boolean variables $x_{i(1,1)}, \ldots, x_{i(1,p_1)}$ as constants in order to make $\Delta_{e_i}(g)(x)=b$. Let $v_{(1,b)}\in\{0, 1\}^{p_1}$ be the binary vector such that $\Delta_{e_i}(g)\,(x)=b$ for any $x\in\{0, 1\}^n$ with $(x_{i(1,1)}, \ldots, x_{i(1,p_1)})=v_{(1,b)}$. Define

$$H_{(1,b)}=\{x\in\{0, 1\}^n: (x_{i(1,1)}, \ldots, x_{i(1,p_1)})=v_{(1,b)}\}$$

and $g_{(1,b)}\doteq g|_{H(1,b)}$ for $b\in\{0, 1\}$. By Lemma 3, there exists a bit b_1 such that $\|\hat{g}^{\pm}_{(1,b_1)}\|_1\leq \|\hat{g}^{\pm}\|_1/2$. Since $g|_{H(1,b)}$ is also a read-k polynomial, we repeat the above procedure on $g_{(1,b)}$. Then there exist a bit b_2, a set of Boolean variables $x_{i(2,1)}, \ldots, x_{i(1,p_2)}$, a vector $v_{(2,b_2)}\in\{0, 1\}^{p_2}$ with $p_2\leq k(d-1)$, and an affine subspace

$$H_{(2,b_2)}=\{x\in\{0, 1\}^n: (x_{i(2,1)}, \ldots, x_{i(1,p_2)})=v_{(2,b_2)}\}$$

such that the function $g_{(2,\, b_2)}$ defined by $g_{(2,\, b_2)} \doteq g_{(1,\, b)}|_{H(1,\, b_2)}$ satisfies that $||\hat{g}^{\pm}_{(2,\, b_2)}||_1 \le ||\hat{g}^{\pm}_{(1,\, b_1)}||_1/2$. Repeat the same procedure q times where q= $\log(||\hat{g}^{\pm}||_1)$. Then we obtain a series of functions $\{\hat{g}^{\pm}_{(i,\, b_i)}: i \in [q]\}$ such that $||\hat{g}^{\pm}_{(i+1,\, b_{i+1})}||_1 \le ||\hat{g}^{\pm}_{(i,\, b_i)}||_1/2$ and $||\hat{g}^{\pm}_{(q,\, b_q)}||_1 \le 1$. Thus, we conclude that $g_{(q,\, b_q)}$ is a linear function by Fact 3. Therefore, we only need at most $k(d-1)\log(||\hat{g}^{\pm}||_1)+1$ linear restrictions to set G constant. This implies that

$$C_{\oplus,\min}(g) = k(d-1)\log((||\hat{g}||_1)+1 = O(dk\log((||\hat{g}||_1))$$

where the last equality holds by Fact 2. ❑

For small k, we have the following corollaries.

Corollary 1: If g is a read-k polynomial with $k \le \log^c((||\hat{g}||_1)$ for a fixed constant c and with F_2-degree d, then $C_{\oplus,\min}(g) = O(d\log^{c+1}((||\hat{g}||_1))$ and $D^{cc}(g \circ \oplus) = O(d^2\log^{c+1}(||\hat{g}||_1))$.

Proof: Since $k \le \log^c((||\hat{g}||_1)$, we have $C_{\oplus,\min}(g) = O(d\log^{c+1}(||\hat{g}||_1))$ by Theorem 1. Now, by Lemma 4, $D^{cc}(g \circ \oplus) = O(d^2\log^{c+1}(||\hat{g}||_1))$. ❑

Corollary 2: If f is a read-k polynomial with $k \le \log^c((||\hat{g}||_0)$ for some constant c and G is the XOR function with respect to g, then $D_{\oplus}(g) = \log^{O(1)}(||\hat{g}||_0)$ and $D^{cc}(G) = \log^{O(1)}(\mathbf{rank}(M_G))$.

Proof: By Theorem 1, Facts 1 and 4, we obtain that $C_{\oplus,\min}(g) = O(\log^{c+2}((||\hat{g}||_0))$. Now, by Lemma 4, Facts 4 and 5, $D^{cc}(G) = \log^{c+3}(||\hat{g}||_0)) = O(\log^{c+3}(\mathbf{rank}(M_G)))$. ❑

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