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Optimality Conditions for Fuzzy Programming Problems with Differentiable Fuzzy Objective Mappings

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In this paper, under the condition of D-differentiation, we consider the fuzzy programming problem with the general fuzzy mapping (non-convex) as the objective mapping. By discussing the characteristics of the optimal solution of unconstrained fuzzy programming, we give the KKT condition of the optimal solution of more general fuzzy programming with real value function as the constrained condition, and some test examples. Meanwhile, we discuss the optimal condition of a special class of fuzzy programming problem with the real-valued concave function as the constrained condition and the convex fuzzy mapping as the objective mapping.

Keywords: fuzzy mapping, D-differentiability, fuzzy programming, constrained condition, optimal condition

1. INTRODUCTION

Fuzzy mapping is a function (fuzzy value function) whose value is fuzzy number, and is an important part of fuzzy analysis. Optimization Theory, an important branch of Mathematics, has widely application. However, the parameters of many mathematics programming are uncertain during the process of mathematical modeling. So it's very important that the fuzzy programming is studied [1-4]. With the deepening research and the development of fuzzy programming, some classical mathematics methods will be generalized and applied to the study of fuzzy programming. Therefore, the differentiation and sub-differentiation of fuzzy mapping and their applications in the fuzzy programming are discussed.

There are many discussions on the differentiation of fuzzy mapping based on different background. The most common used are the generalization of Hakuhara derivative for set-valued function (denoted as H-derivative) and the derivative of fuzzy mapping (denoted as L-derivative), which are defined by Goetschel-Voxmann [5] and Puri-Ralescu [6] and by Buckley-Feuring [7, 8] using the left and right endpoint functions of the horizontal interception set, respectively. Motial Panigrahi *et al.* [9] generalized the differentiation of fuzzy mapping [7, 8] to the situation of multivariable, and obtained the KKT condition of the optimal solution of fuzzy programming with L-differentiable convex fuzzy objective mapping. Hsien-Chung Wu [10, 11] further discussed the H-differentiability of fuzzy mapping and its application in fuzzy programming, and obtained the characters of H-differentiable fuzzy mapping, the optimal condition of saddle point

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and the KKT condition of the optimal solution of H-differentiable convex (or generalized convex) fuzzy objective programming. Bede and Stefanini [12] proposed the generalized differentiability of fuzzy mapping and gave its application. Chalco-Cano [13, 14] provided some new description of generalized differentiable fuzzy mapping and obtained the KKT condition of the optimal solution of fuzzy programming with generalized differentiable convex (or generalized convex) fuzzy objective mapping. Shexiang Hai *et al.* [15] discussed several kinds of generalized convex fuzzy programming under the condition of generalized differentiation and obtained the sufficient condition of optimal solution. Wang and Wu [16] proposed a kind of differentiation of fuzzy mapping (denoted as D-differentiation) and the concepts of the sub-differentiation, gradient and sub-gradient in order to avoid the difficulty brought by H-difference, applied them to the convex fuzzy programming and obtained the characters of solution. Bao *et al.* [17] further studied the D-differentiability of fuzzy mapping, obtained some new descriptions of D-differentiation and gave its application. Zhang *et al.* [18] proposed the concepts of sub-gradient, sub-differentiation and differentiation of convex fuzzy mapping from the point of view of convex analysis, studied the extremum of convex fuzzy mapping and obtained the sufficient/necessary condition of the existence of extremum for convex fuzzy mapping.

Although many results on the fuzzy programming have been obtained, there are still a large number of unexplored problems in this field. Especially, up to now, there are not the research results on the KKT condition of the optimal solution for fuzzy programming problems with the general fuzzy mapping (non-convex) as the objective mapping. There are few results on the optimal conditions of fuzzy programming problems with D-differentiable fuzzy objective mapping. In this paper, under the condition of D-differentiation, we consider the optimal conditions of more general fuzzy programming problem with the general fuzzy mapping (non-convex) as the objective mapping. In this sense, it can be said that the results of this paper are the generalization of some results in the references [9, 11, 14-16, 18]. In Section 2, the gradient of D-differentiable fuzzy mapping and the new description of D-differentiable convex fuzzy mapping are given. In Section 3, the characteristics of the optimal solution of fuzzy programming problem with the D-differentiable fuzzy objective mapping and the unconstrained condition are discussed. In Section 4, the KKT condition of the optimal solution of fuzzy programming problem with D-differentiable fuzzy objective mapping and the constrained condition is discussed. Meanwhile, the optimal condition of a special class of fuzzy programming problem with the real-valued concave function as the constrained condition and the convex fuzzy mapping as the objective mapping is also discussed.

2. PRELIMINARIES

First, we quote some notations, basic definitions and operations about fuzzy number [9, 16, 17].

Let R be the set of all real numbers. A fuzzy set u:$R\to[0, 1]$ in R is called a fuzzy number if u is normal and upper semi-continuous, and its support set is compact set.

We will denote $\mathcal{F}$ as the set of fuzzy numbers and call it the space of fuzzy numbers. It is clear that for any $r \in R$, the fuzzy number $\hat{r}$ can be defined by

$$\hat{r}(t) = \begin{cases} 1, t = r, \\ 0, t \neq r, \end{cases}$$

for any $t \in R$. For $r \in [0,1]$, the r-level set of fuzzy number u is a nonempty bounded closed interval $[u]_r = [u_*(r), u^*(r)]$. Thus the parameter expression of a fuzzy number u can be represented as $u = \{(u_*(r), u^*(r), r)|0 \leq r \leq 1\}$.

For $u, v \in \mathcal{F}$ and $\lambda \in R(\lambda \geq 0)$, the addition, multiplication and scalar product can be represented respectively as below:

$$u + v = \{(u_*(r) + v_*(r), u^*(r) + v^*(r), r)|0 \leq r \leq 1\},$$
$$u \cdot v = \{(uv)_*(r), (uv)^*(r)+v^*(r), r)|0 \leq r \leq 1\},$$
$$\lambda u = \{(\lambda u_*(r), \lambda u^*(r), r)|0 \leq r \leq 1\},$$

where

$$(u + v)_*(r) = u_*(r) + v_*(r), (u + v)^*(r) = u^*(r) + v^*(r),$$
$$(uv)_*(r) = \min\{u_*(r)v_*(r), u_*(r)v^*(r), u^*(r)v_*(r), u^*(r)v^*(r)\},$$
$$(uv)_*(r) = \max\{u_*(r)v_*(r), u_*(r)v^*(r), u^*(r)v_*(r), u^*(r)v^*(r)\}.$$

Definition 2.1 [9]: For $u, v \in \mathcal{F}$, we say that

(i) $u \leq v$ if $u^*(r) \leq v^*(r)$ and $u_*(r) \leq v_*(r)$ for each $r \in [0,1]$;
(ii) $u = v$ if $u \leq v$ and $u \geq v$;
(iii) $u < v$ if $u \leq v$ and there exists $r_0 \in [0,1]$ such that $u_*(r_0) < v_*(r_0)$ or $u^*(r_0) < v^*(r_0)$.

Given $u, v \in \mathcal{F}$, we define the distance between u and v by

$$D(u,v) = \sup_{r \in [0,1]} \max\left\{\left|u_*(r) - v_*(r)\right|, \left|u^*(r) - v^*(r)\right|\right\}.$$

Then $(\mathcal{F}, D)$ is a complete metric space and satisfies

$$D(u + w, v + w) = D(u, v), D(\lambda u, \lambda v) = |\lambda|D(u, v), \text{ for any } u, v, w \in \mathcal{F} \text{ and } \lambda \in R.$$

For $u_i \in \mathcal{F}(i = 1, 2, \ldots, n)$, we define $u = (u_1, u_2, \ldots, u_n)$ as n-dimensional fuzzy vector in $\mathcal{F}$. The set of all n-dimensional fuzzy vector is denoted as $\mathcal{F}^n$. For $x, y \in R^n$, $d(x, y)$ denotes the Euclidean metric between x and y.

For $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n) \in \mathcal{F}^n$ and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in R^n$, we define:

(i) $u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$, $\lambda u = (\lambda u_1, \lambda u_2, \ldots, \lambda u_n)(\lambda \geq 0)$,
(ii) $u = v \Leftrightarrow u_i = v_i(i = 1, 2, \ldots, n)$,
(iii) $< u, x >= \sum_{i=1}^{n} x_i u_i \left(x_i \geq 0, i = 1, 2, \cdots, n\right), < x, y >= \sum_{i=1}^{n} x_i y_i$,

(iv) $\begin{cases} u_*(r) = (u_{1*}(r), u_{2*}(r), \cdots, u_{n*}(r)) \\ u^*(r) = (u_1^*(r), u_2^*(r), \cdots, u_n^*(r)) \end{cases}$, $\begin{cases} < u, x >^* \left(r\right) =< u^*\left(r\right), x >= \sum_{i=1}^{n} x_i u_i^*\left(r\right) \\ < u, x >_* \left(r\right) =< u_*\left(r\right), x >= \sum_{i=1}^{n} x_i u_{i*}\left(r\right) \end{cases}$

for any $r\in[0, 1]$.

In this paper, a mapping $F:M\to\mathcal{F}$ is said to be a fuzzy mapping (fuzzy-valued function), where M is a nonempty subset of R^n and $\mathcal{F}$ is the space of fuzzy numbers. According to the parametric expression of fuzzy number, that fuzzy mapping can be expressed as:

$$F(x) = \{(F(x)_*(r), F(x)^*(r), r)|0 \le r \le 1\}.$$

Where $F(x)_*(r)$ and $F(x)^*(r)$ are real-valued functions defined on M.

Definition 2.2 [16]: Let $F:M\to\mathcal{F}$ be a fuzzy mapping, $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)\in R^n$. If there exists a $u = (u_1, u_2, \ldots, u_n) \in \mathcal{F}^n$ such that

$$\lim_{x\to x^0} D\left(F(x) + \sum_{x_i<x_i^0} \left|x_i - x_i^0\right| u_i, F(x^0) + \sum_{x_i\ge x_i^0} \left|x_i - x_i^0\right| u_i \right) \Big/ d(x, x^0) = 0. \tag{1}$$

then we say that F is differentiable at x^0, and call $(u_1, u_2, \ldots, u_n)$ (denoted by $\nabla F(x^0) = (u_1, u_2, \ldots, u_n)$) the gradient of F at x^0. Let

$$(x - x^0)^+ = ((x_1 - x_1^0)^+, (x_2 - x_2^0)^+, \ldots, (x_n - x_n^0)^+),$$
$$(x - x^0)^- = ((x_1 - x_1^0)^-, (x_2 - x_2^0)^-, \ldots, (x_n - x_n^0)^-),$$

where

$$(x_i - x_i^0)^+ = \begin{cases} x_i - x_i^0, & x_i \ge x_i^0 \\ 0, & x_i < x_i^0 \end{cases}, \quad (x_i - x_i^0)^- = \begin{cases} x_i^0 - x_i, & x_i \le x_i^0 \\ 0, & x_i > x_i^0 \end{cases} \quad \text{for each } i = 1, 2, \ldots, n.$$

Then

(i) $x - x^0 = (x - x^0)^+ - (x - x^0)^-$, $\lambda(x - x^0)^+ = [\lambda(x - x^0)]^+$, $\lambda(x - x^0)^- = [\lambda(x - x^0)]^-(\lambda\ge 0)$.

(ii) Formula (1) can be rewritten as Formula (2)

$$\lim_{x\to x^0} D\left(F(x) + \left\langle u, (x - x^0)^-\right\rangle, F(x^0) + \left\langle u, (x - x^0)^+\right\rangle\right) \Big/ d(x, x^0) = 0. \tag{2}$$

Remark 2.1: In this paper, the differentiation given in Definition 2.2 is called D-differentiation.

Remark 2.2: By the relevant conclusions in [16, 17], we know that H-differentiable fuzzy mappings must be D-differentiable. The following Example 2.1 shows that the converse is not necessarily true.

Example 2.1: Let $F(x) = \{(x^2 + r, x^2 - r + 4, r)|0 \le r \le 1\}$ be a fuzzy mapping defined on $[-1, 1]$. Then for $x = 0\in[-1, 1]$, we have

$$\lim_{x\to 0^+} D\left(F(x), F(0) + (x - 0)\cdot\hat{0}\right) \Big/ |x - 0| = 0,$$
$$\lim_{x\to 0^-} D\left(F(x) + (0 - x)\cdot\hat{0}, F(0)\right) \Big/ |x - 0| = 0.$$

Therefore, F is D-differentiable at $x = 0$, and $\nabla F(0) = (\hat{0})$.

On the other hand, we show that F is not H-differentiable at $x = 0$. Suppose F is H-differentiable at $x = 0$, then there exists $\delta > 0$ such that the H-difference $F(0+h) - F(0)$ exists, *i.e.*, $F(0 + h) - F(0) \in \mathcal{F}$ for any $h \in (0, \delta)$. So $(F(0 + h) - F(0))_*(r) \le (F(0 + h) - F(0))^*(r)$, *i.e.*, $F(0+h)_*(r) - F(0)_*(r) \le F(0+h)^*(r) - F(0)^*(r)$ for any $h \in (0, \delta)$.

Therefore, we have

$$h^2 + r - (0^2 + r) \le h^3 - r + 4 - (0^3 - r + 4).$$

Hence, $h^2 \le h^3$ for any $h \in (0,\delta)$. Thus, we obtain wrong conclusion $h \ge 1$ for any $h \in (0, \delta)$. So F is not H-differentiable at $x = 0$.

Definition 2.3 [16]**:** A fuzzy mapping $F{:}M \to \mathcal{F}$ defined on a convex subset M in R^n is convex if and only if

$$F(\lambda x + (1 - \lambda)y) \le \lambda F(x) + (1 - \lambda)F(y)$$

for any $x, y \in M$ and $\lambda \in [0,1]$.

The convexity of F is linked to convexity of its endpoint functions. In fact, F is convex if and only if $F(x)^*(r)$ and $F(x)_*(r)$ are convex functions for any $r \in [0,1]$.

Lemma 2.1: [19] (Gordan Theorem). Let A be an $m \times n$ matrix. Then $A\overline{x} < 0$ has solutions if and only if there not exists nonzero and non-negative $y \in R^n (y \ne 0)$ such that $A^T y = 0$.

3. D-DIFFERENTIABILITY OF FUZZY MAPPING

In this section, we give the gradient of D-differentiable fuzzy mappings and a new description of D-differentiable convex fuzzy mappings.

Theorem 3.1: Let $F{:}M \to \mathcal{F}$ be a fuzzy mapping,

$$x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \operatorname{int} M,\ u = (u_1, u_2, \ldots, u_n) \in \mathcal{F}^n.$$

Then

$$\nabla F(x^0) = (u_1, u_2, \cdots, u_n) \Leftrightarrow \begin{cases} \nabla F(x^0)_*(r) = (u_{1*}(r), u_{2*}(r), \cdots, u_{n*}(r)) \\ \nabla F(x^0)^*(r) = (u_1^*(r), u_2^*(r), \cdots, u_n^*(r)) \end{cases}$$

for any $r \in [0,1]$.

Proof: Let $\nabla F(x^0) = (u_1, u_2, \ldots, u_n)$. According to Definition 2.2, we have

$$\lim_{x \to x^0} D\Big(F(x) + \big\langle u, (x - x^0)^- \big\rangle, F(x^0) + \big\langle u, (x - x^0)^+ \big\rangle\Big) \Big/ d(x, x^0) = 0 .$$

$$\Leftrightarrow \lim_{x \to x^0} \sup_{r \in [0,1]} \max \left\{ \left| F(x)_*(r) + \sum_{i=1}^n (x_i - x_i^0)^- u_{i*}(r) - F(x^0)_*(r) - \sum_{i=1}^n (x_i - x_i^0)^+ u_{i*}(r) \right| \Big/ d(x, x^0), \right.$$
$$\left. \left| F(x)^*(r) + \sum_{i=1}^n (x_i - x_i^0)^- u_i^*(r) - F(x^0)^*(r) - \sum_{i=1}^n (x_i - x_i^0)^+ u_i^*(r) \right| \Big/ d(x, x^0) \right\} = 0 .$$

$$\Leftrightarrow \begin{cases} \lim_{x\to x^0} \left| F(x)_*(r) - F(x^0)_*(r) - \sum_{i=1}^{n}(x_i - x_i^0)u_{i*}(r) \right| \Big/ d(x,x^0) = 0 \\ \lim_{x\to x^0} \left| F(x)^*(r) - F(x^0)^*(r) - \sum_{i=1}^{n}(x_i - x_i^0)u_i^*(r) \right| \Big/ d(x,x^0) = 0 \end{cases} \text{for any } r\in[0,1].$$

$$\Leftrightarrow \begin{cases} F(x)_*(r) - F(x^0)_*(r) - \sum_{i=1}^{n}(x_i - x_i^0)u_{i*}(r) = o\left(d(x,x^0)\right) \\ F(x)^*(r) - F(x^0)^*(r) - \sum_{i=1}^{n}(x_i - x_i^0)u_i^*(r) = o\left(d(x,x^0)\right) \end{cases} \text{for any } r\in[0,1].$$

$$\Leftrightarrow \begin{cases} \nabla F(x^0)_*(r) = (u_{1*}(r), u_{2*}(r), \cdots, u_{n*}(r)) \\ \nabla F(x^0)^*(r) = (u_1^*(r), u_2^*(r), \cdots, u_n^*(r)) \end{cases} \text{for any } r\in[0,1].$$

Corollary 3.1: Let $F:M\to\mathcal{F}$ be a fuzzy mapping. If F is D-differentiable at $x^0\in M$, then

$$F(x)+\langle\nabla F(x^0), (x-x^0)^-\rangle = F(x^0)+\langle\nabla F(x^0), (x-x^0)^-\rangle + o(\|x-x^0\|)$$

where

$$[o(\|x-x^0\|)]^r = [o(\|x-x^0\|), o(\|x^0-x\|)] \text{ for any } r\in[0,1].$$

***Proof*:** According to Definition 2.2, if F is D-differentiable at $x^0 \in M$, then there exists $u = (u_1, u_2, \ldots, u_n)\in\mathcal{F}^n$ such that

$$\lim_{x\to x^0} D\left(F(x)+\left\langle u,(x-x^0)^-\right\rangle, F(x^0)+\left\langle u,(x-x^0)^+\right\rangle\right)\Big/ d(x,x^0) = 0.$$

From the proof of Theorem 3.1, for any $r\in[0,1]$, we have

$$F(x)_*(r)+\langle u_*(r), (x-x^0)^-\rangle = F(x^0)_*(r)+\langle u_*(r), (x-x^0)^+\rangle + o(d(x, x^0)),$$
$$F(x)^*(r)+\langle u^*(r), (x-x^0)^-\rangle = F(x^0)^*(r)+\langle u^*(r), (x-x^0)^+\rangle + o(d(x, x^0)).$$

This implies that

$$F(x)+\langle\nabla F(x^0), (x-x^0)^-\rangle = F(x^0)+\langle\nabla F(x^0), (x-x^0)^+\rangle + o((x-x^0)).$$

Theorem 3.2: Let M be a convex open set in R^n. A fuzzy mapping $F:M\to\mathcal{F}$ which is D-differentiable is a convex fuzzy mapping if and only if $F(x)+\langle\nabla F(y), (x-y)^-\rangle \geq F(y)+\langle\nabla F(y), (x-y)^+\rangle$ for any $x, y\in M$.

***Proof*:** Necessary. Let $F:M\to\mathcal{F}$ be a D-differentiable convex fuzzy mapping, and the gradient of F at $x\in M$ is denoted as $\nabla F(x) = (u_1, u_2, \ldots, u_n)\in\mathcal{F}^n$.

According to Definition 2.2, we can know that $F(x)_*(r)$ and $F(x)^*(r)$ are both differentiable convex real valued functions on M for any $r\in[0,1]$. And by Theorem 3.1 we have $\nabla F(x)_*(r) = (u_{1*}(r), u_{2*}(r), \ldots, u_{n*}(r))$, $\nabla F(x)^*(r) = (u_1^*(r), u_2^*(r), \ldots, u_n^*(r))$ for any $r\in[0,1]$.

Thus, we can obtain by the properties of convex real-valued function that $F(x)_*(r) \geq F(y)_*(r) + \langle\nabla F(x)_*(r), x-y\rangle$, $F(x)^*(r) \geq F(y)^*(r) + \langle\nabla F(x)^*(r), x-y\rangle$ for any for any $r\in[0,1]$.

This implies that

$$\begin{cases} F(x)_*(r)+\left\langle \nabla F(x)_*(r),(x-y)^-\right\rangle \geq F(y)_*(r)+\left\langle \nabla F(x)_*(r),(x-y)^+\right\rangle \\ F(x)^*(r)+\left\langle \nabla F(x)^*(r),(x-y)^-\right\rangle \geq F(y)^*(r)+\left\langle \nabla F(x)^*(r),(x-y)^+\right\rangle \end{cases}$$

for any for any $r\in[0,1]$. Therefore, $F(x)+\langle\nabla F(y), (x-y)^-\rangle \geq F(y)+\langle\nabla F(y), (x-y)^+\rangle$.

Sufficiency. Supposing that

$$F(x^2)+\langle\nabla F(x^1), (x^2-x^1)^-\rangle \geq F(x^1)+\langle\nabla F(x^1), (x^2-x^1)^+\rangle$$

for any x^1, $x^2\in M$ and for $\lambda\in(0,1)$, taking $y = \lambda x^1+(1-\lambda)x^2$, then $y\in M$. Thus for x^1, x^2, $y\in M$, we have

$$F(x^1)+\left\langle \nabla F(y),(x^1-y)^-\right\rangle \geq F(y)+\left\langle \nabla F(y),(x^1-y)^+\right\rangle, \tag{3}$$

$$F(x^2)+\left\langle \nabla F(y),(x^2-y)^-\right\rangle \geq F(y)+\left\langle \nabla F(y),(x^2-y)^+\right\rangle. \tag{4}$$

So for any $r\in[0,1]$, by Eqs. (3) and (4), we have

$$F(x^1)_*(r)\geq F(y)_*(r)+\left\langle \nabla F(y)_*(r),x^1-y\right\rangle, \tag{5}$$

$$F(x^1)^*(r)\geq F(y)^*(r)+\left\langle \nabla F(y)^*(r),x^1-y\right\rangle. \tag{6}$$

And

$$F(x^2)_*(r)\geq F(y)_*(r)+\left\langle \nabla F(y)_*(r),x^2-y\right\rangle, \tag{7}$$

$$F(x^2)^*(r)\geq F(y)^*(r)+\left\langle \nabla F(y)^*(r),x^2-y\right\rangle. \tag{8}$$

Hence, considering the sum of formula (5) multiplied by λ and formula (7) multiplied by $(1-\lambda)$, we have

$$\lambda F(x^1)_*(r)+(1-\lambda)F(x^2)_*(r)\geq F(y)_*(r)+\left\langle \nabla F(y)_*(r),\lambda x^1+(1-\lambda)x^2-y\right\rangle. \tag{9}$$

Similarly, considering the sum of Eq. (6) multiplied by λ and Eq. (8) multiplied by $(1-\lambda)$, it implies that

$$\begin{aligned} &\lambda F(x^1)_*(r)+(1-\lambda)F(x^2)_*(r)\geq F(y)_*(r)+\left\langle \nabla F(y)_*(r),\lambda x^1+(1-\lambda)x^2-y\right\rangle \\ &=F(y)_*(r)=F\left(\lambda x^1+(1-\lambda)x^2\right)_*(r). \end{aligned} \tag{10}$$

According to Eqs. (9) and (10), we have

$$\lambda F(x^1)+(1-\lambda)F(x^2)\geq F(\lambda x+(1-\lambda)y),$$

for any x^1, $x^2\in M$ and $\lambda\in[0,1]$, which implies that F is convex fuzzy mapping.

4. OPTIMALITY CONDITIONS FOR UNCONSTRAINED FUZZY PROGRAMMING

In this section, we give the characteristics of the optimal solution of unconstrained fuzzy programming.

Let F:$M\to\mathcal{F}$ be a fuzzy mapping, then the following problem

(FP) Minimize $F(x)$,
Subject to $x\in M$,

is called unconstrained fuzzy programming problem, where M is called feasible set, and point $x\in M$ is called a feasible solution.

Since "$\le$" and "$<$" are both partial ordering on $\mathcal{F}$, we may quote some concept of solution in multi-objective programming problems [9, 11, 14].

If $\bar{x}\in M$ and there not exists $x\in M(x\neq \bar{x})$ such that $F(x)\le F(\bar{x})$, then we call $\bar{x}$ the global optimal solution of fuzzy programming problem (FP) on M. If there exists an ε-neighborhood $N(\bar{x}, \varepsilon)$ around $\bar{x}$ for some $\varepsilon>0$ such that there not exists $x\in N(\bar{x}, \varepsilon)\cap M(x\neq \bar{x})$ such that $F(x)\le F(\bar{x})$, then we call $\bar{x}$ a local optimal solution of fuzzy programming problem (FP) on M.

Theorem 4.1: Supposing that fuzzy mapping F:$M\to\mathcal{F}$ is D-differentiable at $\bar{x}\in M$. If there exists the direction $d\in R^n$ such that $\langle\nabla F(\bar{x}), d^+\rangle<\langle\nabla F(\bar{x}), d^-\rangle$. Then there exists $\delta>0$ such that $F(\bar{x}+\lambda d)<F(\bar{x})$, for any $\lambda\in(0, \delta)$.

***Proof*:** Let F be D-differentiable at $\bar{x}$. According to Corollary 3.1, we have

$$F(\bar{x}+\lambda d)+\left\langle\nabla F(\bar{x}),(\lambda d)^-\right\rangle=F(\bar{x})+\left\langle\nabla F(\bar{x}),(\lambda d)^+\right\rangle+o(\|\lambda d\|).$$

So for $r\in[0,1]$, we have

$$F(\bar{x}+\lambda d)_*(r)+\lambda\left\langle\nabla F(\bar{x})_*(r),d^-\right\rangle=F(\bar{x})_*(r)+\lambda\left\langle\nabla F(x)_*(r),d^+\right\rangle+o(\|\lambda d\|),$$
$$F(\bar{x}+\lambda d)^*(r)+\lambda\left\langle\nabla F(\bar{x})^*(r),d^-\right\rangle=F(\bar{x})^*(r)+\lambda\left\langle\nabla F(x)^*(r),d^+\right\rangle+o(\|\lambda d\|).$$

That is

$$F(\bar{x}+\lambda d)_*(r)=F(\bar{x})_*(r)+\lambda\left[\left\langle\nabla F(x)_*(r),d\right\rangle+o(\|\lambda d\|)/\lambda\right], \tag{11}$$
$$F(\bar{x}+\lambda d)^*(r)=F(\bar{x})^*(r)+\lambda\left[\left\langle\nabla F(x)^*(r),d\right\rangle+o(\|\lambda d\|)/\lambda\right]. \tag{12}$$

According to $\langle\nabla F(\bar{x}), d^+\rangle<\langle\nabla F(\bar{x}), d^-\rangle$, we have

$$\left\langle\nabla F(\bar{x})_*(r),d^+\right\rangle-\left\langle\nabla F(\bar{x})_*(r),d^-\right\rangle=\left\langle\nabla F(\bar{x})_*(r),d^+-d^-\right\rangle=\left\langle\nabla F(\bar{x})_*(r),d\right\rangle\le 0.$$

Similarly, it implies that $\langle\nabla F(\bar{x})^*(r), d^+\rangle\le 0$ and there exists $r_0\in[0,1]$ which allows

$$\left\langle\nabla F(\bar{x})^*(r_0),d\right\rangle<0 \quad\text{or}\quad \left\langle\nabla F(\bar{x})^*(r_0),d\right\rangle<0.$$

Again by $\lim_{\lambda\to 0}o(\|\lambda d\|)/\lambda=0$, there exists $\delta>0$ such that

$$\lambda\left[\left\langle\nabla F(x)^*(r),d\right\rangle+o(\|\lambda d\|)/\lambda\right]\le 0,$$
$$\lambda\left[\left\langle\nabla F(x)_*(r),d\right\rangle+o(\|\lambda d\|)/\lambda\right]\le 0.$$

for any $\lambda\in[0,\delta]$ and $r\in[0,1]$. And there exists $r_0\in[0,1]$ such that

$$\lambda\Big[\big\langle\nabla F(x)^*(r_0),d\big\rangle+o\big(\|\lambda d\|\big)/\lambda\Big]<0 \quad\text{or}\quad \lambda\Big[\big\langle\nabla F(x)_*(r_0),d\big\rangle+o\big(\|\lambda d\|\big)/\lambda\Big]<0$$

for any $\lambda\in[0,\delta]$. Thus, according to Eqs. (11) and (12), for $r\in[0,1]$, we have

$$F(\bar{x}+\lambda d)_*(r)\le F(\bar{x})_*(r)\,,$$
$$F(\bar{x}+\lambda d)^*(r)\le F(\bar{x})^*(r)\,.$$

And there exists $r_0\in[0,1]$ such that

$$F(\bar{x}+\lambda d)_*(r_0)<F(\bar{x})_*(r_0)\quad\text{or}\quad F(\bar{x}+\lambda d)^*(r_0)<F(\bar{x})^*(r_0).$$

Therefore, $F(\bar{x}+\lambda d)<F(\bar{x})$, for any $\lambda\in[0,\delta]$.

Theorem 4.2: Let $F{:}M\to\mathcal{F}$ be a D-differentiable convex fuzzy mapping, and $\bar{x}\in M$. If $\nabla F(\bar{x})=0$, then $\bar{x}$ is the global optimal solution.

***Proof*:** Let F be a D-differentiable convex fuzzy mapping, and $\nabla F(\bar{x})=0$. Then for $x\in R^n$,

$$\big\langle\nabla F(\bar{x}),(x-\bar{x})^-\big\rangle=\big\langle\nabla F(\bar{x}),(x-\bar{x})^+\big\rangle=0.$$

According to Theorem 3.2, we have

$$F(x)=F(x)+\big\langle\nabla F(\bar{x}),(x-\bar{x})^-\big\rangle\ge F(\bar{x})+\big\langle\nabla F(\bar{x}),(x-\bar{x})^+\big\rangle=F(\bar{x}).$$

So $\bar{x}$ is the global optimal solution.

Definition 4.1: Let $F{:}M\to\mathcal{F}$ be a fuzzy mapping, $d\in R^n$ be a nonzero vector. We say that d is the descent direction of F at $\bar{x}$ if there exists $\delta>0$ such that

$$F(\bar{x}+\lambda d)<F(\bar{x})\quad\text{for any }\lambda\in(0,\delta).$$

According to Theorem 4.1, d is the descent direction of F at x if F is D-differentiable fuzzy mapping and $\langle\nabla F(x), d^+\rangle<\langle\nabla F(x), d^-\rangle$. Let

$$M_F=\{d|\langle\nabla F(x), d^+\rangle<\langle\nabla F(x), d^-\rangle,\ d\in R^n\text{ and }d\neq 0\}. \tag{13}$$

Then M_F is the descent direction set of F at x.

Definition 4.2: Let $M\subset R^n$ be a closed set, $d\in R^n$ be a nonzero vector, $\bar{x}\in M$. d is the feasible direction of M at $\bar{x}$ if there exists $\delta>0$ such that $\bar{x}+\lambda d\in M$ for any $x\in(0,\delta)$.

Set made up by all of the feasible direction of M at $\bar{x}$ which can be written as

$$D_M=\big\{d\,\big|\,d\neq 0,\exists\delta>0,\forall\lambda\in(0,\delta),\bar{x}+\lambda d\in M\big\}. \tag{14}$$

And D_M is called the cone of feasible directions of M at $\bar{x}$.

Theorem 4.3: Let a fuzzy mapping $F:M\to\mathcal{F}$ be D-differentiable at $\bar{x}$ in a fuzzy programming problem (FP). If $\bar{x}$ is the local optimal solution, then $M_F\cap D_M=\varnothing$.

***Proof*:** Supposing that there exists nonzero vector $d\in M_F\cap D_M$, then $d\in M_F$ and $d\in D_M$. According to Eq. (13), we have $\langle\nabla F(x), d^+\rangle<\langle\nabla F(x), d^-\rangle$. So according to Theorem 4.1, there exists $\delta_1>0$ such that

$$F(\bar{x}+\lambda d)<F(\bar{x}),\ \text{for any}\ \lambda\in(0,\delta_1). \tag{15}$$

Moreover, according to Eq. (14), there exists $\delta_2>0$ such that

$$\bar{x}+\lambda d\in D_M\quad \text{for any}\ \lambda\in(0,\ \delta_2). \tag{16}$$

Taking $\delta=\min\{\delta_1,\ \delta_2\}$, then Eqs. (15) and (16) both hold when $\lambda\in(0,\ \delta)$, which contradicts that $\bar{x}$ is the local optimal point. The proof is complete.

Remark 4.1: By Remark 2.2, the corresponding results still hold when the mapping $F:M\to\mathcal{F}$ in Theorem 4.3 is H-differentiable.

5. OPTIMALITY CONDITIONS FOR CONSTRAINED FUZZY PROGRAMMING

In references [9, 11, 14-16], the (convex) fuzzy programming problem is discussed in which the objective mapping and the constrained function are convex. In this section, under the condition of D-differentiation, we discuss the fuzzy programming problem with the general fuzzy mapping as the objective mapping and the general real-value function as the constrained function (both non-convex), and give the KKT condition of corresponding optimal solution.

Let $F:M\to\mathcal{F}$ be a fuzzy mapping, $G_i : R^n\to R(i = 1, 2, \ldots, m)$ be real valued function. The problem like

(MFP) Minimize $F(x)$,
Subject to $G_i(x)\geq 0$, $i = 1, 2, \ldots, m$,

is called constrained fuzzy programming problem where the set

$$M = \{x|G_i(x)\geq 0,\ i = 1, 2, \ldots, m\}$$

is called feasible set and the point $x\in M$ is called feasible solution.

Constraint conditions satisfying $G_i(\bar{x})>0$ are called inactive constraint at $\bar{x}$. On the other hand, those satisfying $G_i(\bar{x})=0$ are called active constraint at $\bar{x}$.

Let $I = \{i\,|\,G_i(\bar{x})=0\}$. If G_i is differentiable real-valued function, then

$$G_I = \{d\,|\langle\nabla G_i(\bar{x}),\ d\rangle>0,\ i\in I\},$$

can replace the cone of directions in Theorem 4.3.

Theorem 5.1: Suppose that $\bar{x}\in M$. F:$M\to\mathcal{F}$ is D-differentiable at $\bar{x}$, $G_i(i\in I)$, is differentiable at $\bar{x}$ and $G_i(i\notin I)$ is continuous at $\bar{x}$. If $\bar{x}$ is the local optimal solution of fuzzy programming (MFP), then $M_F\cap G_I=\varnothing$.

***Proof*:** According to the proof of Theorem 4.3, we have $M_F\cap D_M=\varnothing$ at point $\bar{x}$.

Next we prove that $G_I\subset D_M$. Suppose that the direction $d\in G_I$. We can obtain that

$$\left\langle \nabla G_i(\bar{x}), d\right\rangle > 0. \tag{17}$$

For any $r\in[0,1]$, taking $H_i(x)_*(r)=-G_i(x)$, $H_i(x)^*(r)=-G_i(x)$. Then the fuzzy mapping H_i:$M\to\mathcal{F}$ is D-differentiable at $\bar{x}$, and

$$\nabla H_i(\bar{x})_*(r)=\nabla H_i(\bar{x})^*(r)=-\nabla G_i(\bar{x})$$

for any $r\in[0,1]$. That is, we have $\nabla H_i(\bar{x})=-\nabla G_i(\bar{x})$. By Eq. (17), it implies that

$$\begin{aligned}&\left\langle \nabla H_i(\bar{x}), d^+\right\rangle-\left\langle \nabla H_i(\bar{x}), d^-\right\rangle=\left\langle -\nabla G_i(\bar{x}), d^+\right\rangle-\left\langle -\nabla G_i(\bar{x}), d^-\right\rangle\\&=-\left\langle \nabla G_i(\bar{x}), d^+-d^-\right\rangle=-\left\langle \nabla G_i(\bar{x}), d\right\rangle<0.\end{aligned}$$

Hence, $\left\langle \nabla H_i(\bar{x}), d^+\right\rangle<\left\langle \nabla H_i(\bar{x}), d^-\right\rangle$. Thus, according to Theorem 4.1, there exists $\delta_1>0$ such that

$$H_i(\bar{x}+\lambda d)<H_i(\bar{x})(i\in I) \quad \text{for each } \lambda\in(0,\delta_1).$$

So for all $i\in I$, we have $G_i(\bar{x}+\lambda d)>G_i(\bar{x})=0$. Since $G_i(\bar{x})>0$ when $i\notin I$, by the continuity of $G_i(i\notin I)$ at $\bar{x}$, there exists $\delta_2>0$ such that

$$G_i(\bar{x}+\lambda d)>0 \quad \text{for each } x\in(0,\ \delta_2).$$

Taking $\delta=\min(\delta_1,\ \delta_2)$, then we can obtain that $G_i(\bar{x}+\lambda d)>0$ for each $i=1, 2, \ldots, m$ and $\lambda\in(0,\lambda)$. So $\bar{x}+\lambda d\in M$. According to Definition 4.2, we have $d\in D_M$. Thus, $G_I\subset D_M$, that is $M_F\cap G_I=\varnothing$.

Theorem 5.2: Suppose that $\bar{x}\in M$. F:$M\to\mathcal{F}$ is D-differentiable at $\bar{x}$, $G_i(i\in I)$ is differentiable at $\bar{x}$ and $G_i(i\notin I)$ is continuous at $\bar{x}$. If $\bar{x}$ is a local optimal solution of problem (MFP), then for any $r\in[0,1]$ there exists non-negative real numbers (no all zero) $w_{0*}(r)$, $w_{i*}(r)(i\in I)$, $w_0^*(r)$, $w_i^*(r)(i\in I)$, such that

$$\begin{aligned}&w_{0*}(r)\nabla F(\bar{x})_*(r)-\sum_{i\in I}w_{i*}(r)\nabla G_i(\bar{x})=0,\\&w_0^*(r)\nabla F(\bar{x})^*(r)-\sum_{i\in I}w_i^*(r)\nabla G_i(\bar{x})=0.\end{aligned}$$

***Proof*:** Suppose $\bar{x}$ is a local optimal solution of (MFP). According to Theorem 5.1, we have $M_F\cap G_I=\varnothing$. So the following inequalities system is unsolvable

$$\begin{cases}-\left\langle \nabla G_i(\bar{x}), d\right\rangle<0\\ \left\langle \nabla F(x)_*(r), d\right\rangle<0\\ \left\langle \nabla F(x)^*(r), d\right\rangle<0\end{cases}$$

for any $r\in[0,1]$. Therefore, for any $r\in[0,1]$, according to Lemma 2.1 there exists non-zero and non-negative vector $w_* = (w_{0*}(r), w_{i*}(r), i\in I)$, $w^* = (w_0^*(r), w_i^*(r), i\in I)$, such that

$$w_{0*}(r)\nabla F(\bar{x})_*(r) - \sum_{i\in I} w_{i*}(r)\nabla G_i(\bar{x}) = 0,$$
$$w_0^*(r)\nabla F(\bar{x})^*(r) - \sum_{i\in I} w_i^*(r)\nabla G_i(\bar{x}) = 0.$$

Theorem 5.3 (KKT Condition): Suppose F is D-differentiable at $\bar{x}$, $G_i(i\in I)$ is differentiable at $\bar{x}$ and $G_i(i\notin I)$ is continuous at $\bar{x}$ in fuzzy programming (MFP). $\{\nabla G_i(\bar{x}) | i\in I\}$ are linearly independent. If $\bar{x}$ is a local optimal solution, then there exists two non-negative real-valued function families $w_{i*}(r)(i\in I)$ and $w_i^*(r)(i\in I)$, defined on [0,1], such that

$$\nabla F(\bar{x})_*(r) - \sum_{i\in I} w_{i*}(r)\nabla G_i(\bar{x}) = 0,$$
$$\nabla F(\bar{x})^*(r) - \sum_{i\in I} w_i^*(r)\nabla G_i(\bar{x}) = 0$$

for any $r\in[0,1]$.

***Proof*:** Let $\bar{x}$ be a local optimal solution of fuzzy programming (MFP). Then According to Theorem 5.2, there are two real number families $w_{0*}(r)$, $\bar{w}_{i*}(r)(i\in I)$ and $w_0^*(r)$, $\bar{w}_i^*(r)(i\in I)$, which are not all zero and non-negative, such that

$$w_{0*}(r)\nabla F(\bar{x})_*(r) - \sum_{i\in I} \bar{w}_{i*}(r)\nabla G_i(\bar{x}) = 0,$$
$$w_0^*(r)\nabla F(\bar{x})^*(r) - \sum_{i\in I} \bar{w}_i^*(r)\nabla G_i(\bar{x}) = 0$$

for any $r\in[0,1]$. Thus, we can obtain that $w_{0*}(r)\neq 0$ and $w_{0*}(r)\neq 0$ for any $r\in[0,1]$ by the linear independence of $\{\nabla G_i(\bar{x}) | i\in I\}$. (Otherwise it implies that $\{\nabla G_i(\bar{x}) | i\in I\}$ are linearly dependent because $\bar{w}_{i*}(r)(i\in I)$ and $\bar{w}_i^*(r)(i\in I)$ are not all zero). Therefore, taking $w_{0*}(r) = \bar{w}_{i*}(r)/w_{0*}(r)$, $w_i^*(r) = \bar{w}_i^*(r)/w_0^*)$ for each $i\in I$, then $w_{i*}(r)(i\in I)$ and $w_i^*(r)(i\in I)$ are two non-negative function families defined on [0,1], and satisfy

$$\nabla F(\bar{x})_*(r) - \sum_{i\in I} w_{i*}(r)\nabla G_i(\bar{x}) = 0\ ,\ \nabla F(\bar{x})^*(r) - \sum_{i\in I} w_i^*(r)\nabla G_i(\bar{x}) = 0 \text{ for any } r\in[0,1].$$

Remark 5.1: In Theorem 5.3, if $G_i(i\notin I)$ is differentiable at $\bar{x}$, then we easily obtain the following the KKT condition of the optimal solution.

$$\begin{cases} \nabla F(\bar{x})_*(r) - \sum_{i=1}^{m} w_{i*}(r)\nabla G_i(\bar{x}) = 0 \\ \nabla F(\bar{x})^*(r) - \sum_{i=1}^{m} w_i^*(r)\nabla G_i(\bar{x}) = 0 \\ w_{i*}(r)G_i(\bar{x}) = 0 \\ w_i^*(r)G_i(\bar{x}) = 0 \\ w_{i*}(r)\geq 0, i=1,2,\cdots,m \\ w_i^*(r)\geq 0, i=1,2,\cdots,m \end{cases}$$

for any $r\in[0,1]$.

Example 5.1: we Consider the following fuzzy programming problem:

$$\begin{cases} \min F(x_1, x_2) = \tilde{x}_1 \cdot \tilde{x}_1 + \tilde{x}_2 \cdot \tilde{x}_2 \\ x_1 + x_2 - 4 \geq 0 \\ x_1 \geq 1, x_2 \geq 1 \end{cases}$$

where

$$\tilde{x}_1 = \left\{(x_1 - (1-r), x_1 + (1-r), r) \middle| 0 \leq r \leq 1\right\}, \quad \tilde{x}_2 = \left\{(x_2 - (1-r), x_2 + (1-r), r) \middle| 0 \leq r \leq 1\right\}$$

are triangular fuzzy numbers. Then through the addition and multiplication of fuzzy numbers, we get

$$F(x_1, x_2) = \left\{\left(\left[x_1 - (1-r)\right]^2 + \left[x_2 - (1-r)\right]^2, \left[x_1 + (1-r)\right]^2 + \left[x_2 + (1-r)\right]^2, r\right) \middle| 0 \leq r \leq 1\right\}.$$

Therefore, according to Theorem 3.1, we can obtain that

$$\nabla F(x_1, x_2) = (\{(2(x_1 - 1 + r), 2(x_1 + (1-r)), r) \mid 0 \leq r \leq 1\}, \{(2(x_2 - 1 + r), 2(x_2 + (1 - r)), r) \mid 0 \leq r \leq 1\}.$$

Let

$$G_1(x_1, x_2) = x_1 + x_2 - 4, \; G_2(x_1, x_2) = x_1 - 1, \; G_3(x_1, x_2) = x_2 - 1,$$

then we have

$$\nabla G_1(x_1, x_2) = (1, 1), \nabla G_1(x_1, x_2) = (1, 0), \nabla G_2(x_1, x_2) = (0, 1).$$

So by Eq. (18), the KKT condition is

$$\begin{cases} \left(2(x_1 - 1 + r), 2(x_2 - 1 + r)\right) - w_{1*}(r)(1,1) - w_{2*}(r)(1,0) - w_{3*}(r)(0,1) = 0 \\ \left(2(x_1 + 1 - r), 2(x_1 + 1 - r)\right) - w_1^*(r)(1,1) - w_2^*(r)(1,0) - w_3^*(r)(0,1) = 0 \\ w_{1*}(r)(x_1 + x_2 - 4) = w_2^*(r)(x_1 + x_2 - 4) = 0 \\ w_{2*}(r)(x_1 - 1) = w_2^*(r)(x_1 - 1) = 0 \\ w_{3*}(r)(x_2 - 1) = w_3^*(r)(x_2 - 1) = 0 \\ w_{i*}(r) \geq 0 \left(i = 1,2,3\right) \\ w_i^*(r) \geq 0 \left(i = 1,2,3\right) \end{cases}$$

for any $r \in [0,1]$. After some algebraic calculations, we easily obtain two non-negative real valued functions defined on [0,1]:

$$w_{1*}(r) = 2{+}2r, w_{2*}(r) = 0, w_{3*}(r) = 0$$

$$w_1^*(r) = 6{+}2r, w_2^*(r) = 0, w_3^*(r) = 0$$

and $\bar{x} = (2, 2)$. Therefore, $\bar{x} = (2, 2)$ is the point which satisfies the KKT conditions.

Theorem 5.4: Assume that the real valued constraint functions G_i(i = 1, 2, …, m) are concave and differentiable, and the fuzzy mapping F is convex on M and D-differentiable at $\bar{x}\in M$. If the KKT condition of the optimal solution of problem (MFP) is true at $\bar{x}$, then $\bar{x}$ is the global optimal solution.

***Proof*:** Suppose that the fuzzy mapping F is convex on M and D-differentiable at $\bar{x}\in M$. According to Theorem 3.2, for any $x\in M$, we get

$$F(x)+\left\langle \nabla F(\bar{x}),(x-\bar{x})^{-}\right\rangle \geq F(\bar{x})+\left\langle \nabla F(\bar{x}),(x-\bar{x})^{+}\right\rangle.$$

Thus for any $r\in(0,1)$, we have

$$F(x)_*(r)+\left\langle \nabla F(\bar{x})_*(r),(x-\bar{x})^{-}\right\rangle \geq F(\bar{x})_*(r)+\left\langle \nabla F(\bar{x})_*(r),(x-\bar{x})^{+}\right\rangle,$$
$$F(x)^*(r)+\left\langle \nabla F(\bar{x})^*(r),(x-\bar{x})^{-}\right\rangle \geq F(\bar{x})^*(r)+\left\langle \nabla F(\bar{x})^*(r),(x-\bar{x})^{+}\right\rangle.$$

So

$$F(x)_*(r)\geq F(\bar{x})_*(r)+\left\langle \nabla F(\bar{x})_*(r),x-\bar{x}\right\rangle, \tag{19}$$
$$F(x)^*(r)\geq F(\bar{x})^*(r)+\left\langle \nabla F(\bar{x})^*(r),x-\bar{x}\right\rangle. \tag{20}$$

Moreover, KKT conditions are true at $\bar{x}$, that is, there are two non-negative real valued function families $w_{i*}(r)(i\in I)$ and $w_i^*(r)(i\in I)$ defined on [0,1], such that

$$\nabla F(\bar{x})_*(r)=\sum_{i\in I} w_{i*}(r)\nabla G_i(\bar{x}), \tag{21}$$
$$\nabla F(\bar{x})^*(r)=\sum_{i\in I} w_i^*(r)\nabla G_i(\bar{x}). \tag{22}$$

Now Eqs. (19)-(22) imply that

$$F(x)^*(r)\geq F(\bar{x})^*(r)+\sum_{i\in I} w_i^*(r)\left\langle \nabla G_i(\bar{x}),x-\bar{x}\right\rangle, \tag{23}$$
$$F(x)_*(r)\geq F(\bar{x})_*(r)+\sum_{i\in I} w_{i*}(r)\left\langle \nabla G_i(\bar{x}),x-\bar{x}\right\rangle. \tag{24}$$

Since G_i are real valued concave functions for i = 1, 2, …, m, $-G_i$ are real valued convex functions for i = 1, 2, …, m. Therefore, for $i\in I$ we have

$$-G_i(x)\geq -G_i(\bar{x})+\left\langle -\nabla G_i(\bar{x}),x-\bar{x}\right\rangle.$$

i.e.,

$$\left\langle \nabla G_i(\bar{x}),x-\bar{x}\right\rangle \geq G_i(x)-G_i(\bar{x}) \text{ for each } i\in I.$$

Thus by $G_i(\bar{x})=0, G_i(x)\geq 0(i\in I)$, we can obtain that

$$\left\langle \nabla G_i(\bar{x}),x-\bar{x}\right\rangle \geq 0(i\in I).$$

Now by Eqs. (23) and (24), it implies that

$$F(x)_*(r)\geq F(\bar{x})_*(r),\quad F(x)^*(r)\geq F(\bar{x})^*(r)$$

for any $r \in [0,1]$. Thus, $F(x) \geq F(\bar{x})$. Hence, $\bar{x}$ is the global optimal solution of problem (MFP).

Remark 5.2: By Remarks 2.2 and 4.1, the corresponding results still hold when the fuzzy mappings $F:M \to \mathcal{F}$ in Theorem 5.3 and Theorem 5.4 are H-differentiable.

Remark 5.3: The KKT conditions of the optimal solution in [9, 11, 14-16] are only applicable to a special class of fuzzy programming (convex fuzzy programming) problems, while the KKT condition of the optimal solution obtained in this paper is applicable to more general fuzzy programming (non-convex fuzzy programming) problems.

6. CONCLUSIONS

In references [9, 11, 14-16], under the conditions of L-differentiation, H-differentiation and generalized differentiation, the (convex) fuzzy programming problem with the convex objective mapping is discussed, and the KKT conditions of corresponding optimal solution are obtained. In this paper, under the condition of D-differentiation, we discuss the fuzzy programming problem with the general fuzzy mapping (non-convex) as the objective mapping. By discussing the characteristics of the optimal solution of unconstrained fuzzy programming, we give the KKT condition of the optimal solution of more general fuzzy programming (non-convex) with real value function as the constrained condition. We also discuss the optimal condition of a special class of fuzzy programming problem with the real-valued concave function as the constrained condition and the convex fuzzy mapping as the objective mapping. The research method in this paper provides a new method for further research on the fuzzy programming problem in which the general fuzzy mapping is the objective mapping. In particular, under the condition of D-differentiation, some results obtained in this paper will provide a good foundation for further studying the KKT condition of the optimal solution of fuzzy programming problem with the fuzzy mapping as the constrained condition.

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