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On the 2-Vertex-Fault Hamiltonicity for Graphs Satisfying Ore's Theorem

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Any undirected and simple graph $G = (V, E)$, where V and E denote the vertex set and the edge set of G, is called Hamiltonian if it contains a cycle that visits each vertex of G exactly once. Ore proved that G is Hamiltonian if $deg_G(u) + deg_G(v) \geq n$ holds for every nonadjacent pair of vertices u and v in V, where n is the total number of distinct vertices of G. Su, Shih, and Kao proved that any graph G satisfying Ore's condition remains Hamiltonian after removing any one vertex $x \in V$ unless G belongs to one of two exceptional families of graphs. This paper proves that $G - \{x, y\}$ is Hamiltonian for any two vertices $x, y \in V$, unless G belongs to one of the eight exceptional families of graphs, denoted by η_i, where $i \in \{1, \ldots, 8\}$.

Keywords: degree, Ore's condition, Hamiltonian, 1-vertex fault Hamiltonian, 2-vertex fault Hamiltonian

1. INTRODUCTION

In this paper, we follow the definitions and notations from [1], and consider undirected and simple graphs only. Let $G = (V, E)$ be a graph with finite vertex set V and edge set $E \subseteq \{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. Let $|G|$ or $|V|$ denote the number of distinct vertices in G, K_n be the complete graph with n vertices, $\overline{K_n}$ be the graph with n isolated vertices, and H_i be a simple graph with i vertices. Two vertices u and v of G are *adjacent* if $(u, v) \in E$. Given a vertex u of G, the *neighborhood* of u, denoted by $N_G(u)$, is the set $\{v \mid (u, v) \in E\} \subseteq V$. The *degree* of u, denoted by $deg_G(u)$, is defined by $deg_G(u) = |N_G(u)|$. The *minimum degree* of G, denoted by $\delta(G)$, that is $\min\{deg_G(u) \mid u \in V(G)\}$; $\sigma_2(G)$ is defined by $\sigma_2(G) = \min\{deg_G(u) + deg_G(v) \mid u$ and v are non-adjacent vertices of $G\}$. Let S be a subgraph of G. $N_S(u)$ and $deg_S(u)$ are defined by $N_S(u) = N_G(u) \cap S$, and $deg_S(u) = |N_S(u)|$. Two edges in a graph G are called *vertex-disjoint-edges*, if the two edges have no common vertex. A *path* in a graph is a single vertex or an ordered list of distinct vertices $v_0, v_1, \ldots, v_k$ such that (v_{i-1}, v_i) is an edge for $1 \leq i \leq k$. The first and the last vertices of a path are its *endpoints*. Let C_m denote a cycle with m vertices, where a *cycle* is a path of at least three vertices among which the first vertex is the same as the last vertex. A path (cycle) is a *Hamiltonian path* (*cycle*) if it traverses all vertices of V exactly once. A Hamiltonian graph is a graph with a Hamiltonian cycle. A non-Hamiltonian graph G is *maximal* if the addition

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of any edge transforms the graph into a Hamiltonian one [2]. The *length* of a path or a cycle is the number of its edges [1]. The subgraph of G induced by S, denoted by $G[S]$, is the subgraph formed by the vertex set S and the edges of G that connect two vertices in S. Specifically, the graph $G[V-S]$ is denoted by $G-S$ and for a vertex v of G, $G-v$ is used to denote $G-\{v\}$.

In addition, for vertices v_i and v_k with $i \le k$, $\langle v_i \nearrow v_k \rangle$ is a path notation used for simplicity to denote $\langle v_i, v_{i+1}, v_{i+2}, \ldots, v_{k-1}, v_k \rangle$, and $\langle v_k \searrow v_i \rangle$ to denote $\langle v_k, v_{k-1}, v_{k-2}, \ldots, v_{i+1}, v_i \rangle$ [3].

A graph G is *connected* if it has a path from u to v for each pair of distinct vertices u, $v \in V(G)$. A *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G-S$ has more than one component. A graph is *k-connected* if every vertex cut has at least k vertices. The *connectivity* of G, denoted by $\kappa(G)$, is the minimum size of a vertex cut. That means $\kappa(G)$ is the maximum k such that G is k-connected. A graph G is Hamiltonian-connected if there exists a Hamiltonian path joining any two different vertices of G.

Theorem 1, a well-known theorem proved by Ore [4], has inspired many studies about Hamiltonian graphs.

Theorem 1: A simple graph $G=(V, E)$ with $|G|=|V|=n \ge 3$ is Hamiltonian if, for each pair of nonadjacent vertices u and v in V, $deg_G(u) + deg_G(v) \ge n$.

Theorem 2: Suppose that G is a graph and u, v are distinct nonadjacent vertices of G with $deg_G(u) + deg_G(v) \ge |G|$. Then G is Hamiltonian if and only if $G+(u, v)$ is Hamiltonian [4].

Theorem 3: Let $G=(V, E)$ be a Hamiltonian graph and S be a subset of V. Then the graph $G-S$ has at most $|S|$ components [1].

Given a graph $G=(V, E)$, $\tilde{E} \subseteq E$, *and* $F \subseteq V \cup E$, we use $G-\tilde{E}$ to denote the subgraph obtained by removing $\tilde{E}$ from G, and $G-F$ to denote the graph obtained by removing F from G, where $V(G-F)=V-F\cap V$ and $E(G-F)=E-\{e \mid e$ is adjacent to any vertex in $F \cap V\} - E \cap F$. Suppose that $G-F$ is Hamiltonian for any $F \subseteq V \cup E$ and $|F| \le k$. Then G is called a *k-fault-Hamiltonian* graph. If $F \subseteq V$ and $|F| \le k$, G is called a *k-vertex-fault-Hamiltonian* graph; if $F \subseteq E$ and $|F| \le k$, G is called a *k-edge-fault-Hamiltonian* graph. It is easy to see that every *k-fault-* (*k-vertex-fault-* or *k-edge fault-*) *Hamiltonian* graph has at least $k+3$ vertices [1]. Moreover, the degree of each vertex in a *k-fault-Hamiltonian* graph is found to be at least $k+2$ [1].

To study Hamiltonian fault-tolerance, we introduce several operations for graphs. Let $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ be two simple graphs. We say that G_1 and G_2 are *disjoint* if they have no vertex in common, and they are *edge-disjoint* if they have no edge in common. The *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is a graph with $V(G_1 \cup G_2)=V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2)=E(G_1) \cup E(G_2)$; if G_1 and G_2 are disjoint, we sometimes denote their union by G_1+G_2, and the union of k copies of G_1 by kG_1. The *join* of disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from G_1+G_2 by joining each vertex of G_1 to each vertex of G_2 [1].

In 1985, Ainouche and Christofides proved the following two theorems [5, 6].

Theorem 4: If G' is a connected graph of order $n' \ge 3$ such that $deg_{G'}(x) + deg_{G'}(y) \ge n'-1$ for each pair of nonadjacent vertices x, y in G', then G' is Hamiltonian or $G' \in \{H_1 \vee (K_h \cup$

K_t), $H_{(n'-1)/2} \vee \overline{K_{(n'+1)/2}}$}.

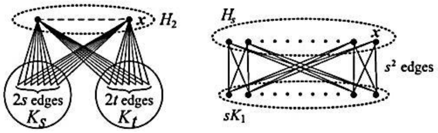
Theorem 5: Let $G'' = (V, E)$ be a 2-connected maximal non-hamiltonian graph of order $n'' \geq \mathbf{5}$. If $deg_{G''}(a)+deg_{G''}(b) \geq |G''| - 2$ for any two non-adjacent vertices a, b, then G'' is isomorphic to one of the following five graphs: $G_1'' = K_{(n''-1)} \vee \overline{K_{(n''+1)/2}}$, n'' is odd; $G_2'' = K_{(n''-2)/2} \vee \overline{K_{(n''+2)/2}}$, n'' is even; $G_3'' = K_{(n''-2)/2} \vee (\overline{K_{(n''-2)/2}} \cup K_2)$, n'' is even; $G_4'' = K_2 \vee (2K_2 \cup K_1)$; $G_5'' = K_2 \vee 3K_2$.

If $\kappa(G) \geq 2$ is added to Theorem 4, then, it can be concluded that G' is Hamiltonian or $G' = H_{(n'-1)/2} \vee \overline{K_{(n'+1)/2}}$.

In 2012, Su, Shih, and Kao proved in the following theorem that a graph G satisfying Ore's condition can be 1-*fault Hamiltonian* except that G belongs to two families of graphs [7].

Theorem 6: Let $G = (V, E)$ be a graph with $|G| = |V| = n \geq 3$. Suppose that $deg_G(u) + deg_G(v) \geq n$ holds for any nonadjacent pair $\{u, v\} \subset V$, then either G is 1-*vertex-fault hamiltonian* or G belongs to one of the two families $\mathcal{G}_1$ and $\mathcal{G}_2$. In addition, G is either 1-*edge-fault hamiltonian* or $G \in \mathcal{G}_1$ with $s \in \{1, 2\}$.

In Theorem 6, the two exceptional families of graphs are:
$\mathcal{G}_1 \equiv \{K_3\} \cup \{H_2 \vee (K_s + K_t) | s + t = n - 2, s \geq 1, t \geq 1\}$, and $\mathcal{G}_2 \equiv \{H_s \vee sK_1 | 2s = n\}$, where H_2 is any simple graph with 2 vertices, H_s is any simple graph with s vertices, as illustrated in Fig. 1.



(a) (b)

Fig. 1. An illustration of graphs of (a) $\{H_2 \vee (K_s + K_t) \,|\, s + t = n - 2, s \geq 1, t \geq 1\}$ in $\mathcal{G}_1$; (b) $\mathcal{G}_2$.

If the condition $\kappa(G) \geq 3$ is added to Theorem 6, and let $|G| = |V| = n \geq 4$, then, either G is 1-*vertex-fault Hamiltonian* or $G \in \mathcal{G}_2$.

In 2013, Zhao pointed out some non-hamiltonian graphs in the following two theorems [8].

Theorem 7: If G'' is a connected graph of order $n'' \geq 3$ such that $deg_{G''}(x) + deg_{G''}(y) \geq n'' - 2$ for each pair of nonadjacent vertices x, y in G'', then either G'' is Hamiltonian or G'' is isomorphic to one of the following nine graphs: (1) $K_{1,3}$; (2) $H_2 \vee 3K_2$; (3) $H_2 \vee (2K_2 \cup K_1)$; (4) $K_h : w : K_t'$; (5) $(H_{(n''-1)/2} \vee \overline{K_{(n''+1)/2}}) - e$; (6) K_1: C_6'; (7) $H_{(n''-1)/2} \vee \overline{K_{(n''+1)/2}}$; (8) $H_{(n''-2)/2} \vee (\overline{K_{(n''-2)/2}} \cup K_2)$; (9) $H_{(n''-2)/2} \vee \overline{K_{(n''+2)/2}}$.

Theorem 8: If G'' is a 2-*connected* graph of order $n'' \geq 9$ such that $deg_{G''}(x)+deg_{G''}(y) \geq n'' - 2$ for each pair of nonadjacent vertices x, y in G'', then G'' is Hamiltonian or $G'' \in \{(H_{(n''-1)/2} \vee \overline{K_{(n''+1)/2}}) - e, H_{(n''-1)/2} \vee \overline{K_{(n''+1)/2}}, H_{(n''-2)/2} \vee (\overline{K_{(n''-2)/2}} \cup K_2), H_{(n''-2)/2} \vee \overline{K_{(n''+2)/2}}\}$.

The graphs (4)-(6) given in Theorem 7 need further discussion. In Theorem 7, according to [8], the notation K_t' in (4) denotes a graph removing some (none, one, or more) *vertex-disjoint edges* of K_t; and the operating notation ":" denotes that edges are added from w to K_h and K'_t as long as $\sigma_2(G'') \geq |G''| - 2$ holds. Apparently, $\kappa(G'') = 1$. See Fig. 2. It is easy to see that $K_h : w : K'_t$ can be replaced by $K'_h : w : K'_t$. In the graph, "e" stands for an edge (v_i, v_k), where $v_i \in H_{(n''-1)/2}$, $v_k \in \overline{K_{(n''+1)/2}}$.

In addition, based on the definition in [8], $V(K_1 : C'_6) = V(K_1) \cup V(C_6)$ with $V(K_1) = \{u\}$, $C_6 = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle$, $V(C'_6) = V(C_6)$, $E(C'_6) = E(C_6) \cup (v_1, v_4) \cup (v_3, v_6) \cup (v_5, v_2)$, and $E(K_1 : C'_6) = E(C'_6) \cup (u, v_6) \cup (u, v_2)$, the graph (6), $G'' = K_1 : C'_6$, can be illustrated by Fig. 3 (a), that is $G'' = \overline{K_3} \vee \overline{K_4} - (v_4, v_7 = u)$. Thus G'' belongs to $H_3 \vee \overline{K_4} - e$, which is (5) in Theorem 7.

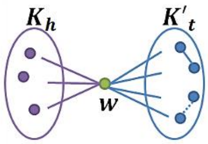
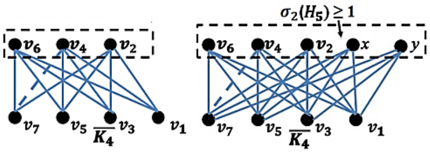


Fig. 2. $G'' = K_h : w : K'_t$.



(a) (b)

Fig. 3. (a) $G'' = (K_1 : C'_6) = \overline{K_3} \vee \overline{K_4} - (v_4, v_7)$. (b) $G \in \eta_3$.

In 2016, we attempted to explore the topic of 2-vertex-fault Hamiltonian graph, and 2-edge-fault Hamiltonian graph. Some preliminary findings are presented in [9]. However, the results there are incomplete, and no formal proof is provided.

In this paper, we aim to find the exceptional families of any 2-*vertex-fault Hamiltonian* graph satisfying the degree-sum condition in Theorem 1. Since G is not 2-*vertex-fault* tolerant (2-*edge-fault* tolerant) when the *vertex-connectivity* of a graph G is equal to or less than 3, we only consider graphs whose vertex connectivities are greater than or equal to 4.

2. MAIN RESULTS

The graph $G_1 : G_2$ is defined to be a graph obtained from $G_1 + G_2$ by connecting some vertices of G_1 to some vertices of G_2, possibly with constraints on how edges are added. For three simple graphs G_1, G_2 and G_3, the notation $G_1 : G_2 : G_3$ is defined to be $G_1 : G_2 : G_3 = (G_1 : G_2) : G_3$. So $G_1 : G_2 : G_3$ is the graph obtained from $G_1 + G_2 + G_3$ by connecting some vertices of G_s to some vertices of G_t, possibly with constraints on how edges are added, where $s, t \in \{1, 2, 3\}$ and $s \neq t$. Therefore, $G_1 + G_2 \subseteq G_1 : G_2 \subseteq G_1 \vee G_2$, and $(G_1 + G_2 + G_3) \subseteq G_1 : G_2 : G_3 \subseteq (G_1 \vee G_2) \vee G_3$. For example, suppose $G_1 : G_2 : G_3 = H_i : x : y$, where H_i is a simple graph with i vertices, and x and y are two vertices not belonging to $V(H_i)$. Then $H_i + x + y \subseteq H_i : x : y \subseteq H_i \vee x \vee y$.

For studying 2-*vertex-fault* Hamiltonian graphs, we introduce the following definition.

Definition 9: Let H_k be any simple graph with k vertices. Define η_i for $1 \leq i \leq 8$ as below.
(1) $\eta_1 = H_4 \vee 3K_2$. See Fig. 4.
(2) $\eta_2 = H_4 \vee (2K_2 \cup K_1)$. See Fig. 5.
(3) $\eta_3 = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}}) - (v_\alpha, v_\theta)$, $v_\alpha \in V(H_{(n+1)/2})$, $v_\theta \in V(\overline{K_{(n-1)/2}})$, n is odd, $n \geq 9$, $\sigma_2(H_{(n+1)/2}) \geq 1$, and $deg_{H_{(n+1)/2}}(v_\alpha) \geq 2$. See Fig. 6 (a).
(4) $\eta_4 = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}})$, n is odd, $n \geq 7$. When $n = 7$, $\kappa(H_4) \geq 1$; when $n \geq 9$, $\sigma_2(H_{(n+1)/2}) \geq$

1. See Figs. 7 and 8.

(5) $\eta_5 = (H_{n/2}) \vee (\overline{K_{(n-4)/2}} \cup K_2)$ where $H_{n/2} \neq \overline{K_{n/2}}$, n is even and $n \geq 8$. See Fig. 9 (a).

(6) $\eta_6 = (H_{n/2}) \vee \overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_\theta)$ for n even and $n \geq 8$, where the complete graph in $(\overline{K_{(n-4)/2}} \cup K_2)$ is with (v_1, v_2), $v_\theta \in (v_1, v_2)$, and $v_\alpha \in H_{n/2}$ with $deg_{H(n/2)}(v_\alpha) \geq 1$. See Fig. 9 (c), Figs. 10 (a), and (b).

(7) $\eta_7 = (H_{n/2}) \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_\theta) - (v_\omega, v_\varepsilon)$ for n even and $n \geq 8$, where the complete graph in $(\overline{K_{(n-4)/2}} \cup K_2)$ is with (v_1, v_2); $v_\theta, v_\varepsilon \in \{v_1, v_2\}$, $v_\theta \neq v_\varepsilon$; $v_\alpha, v_\omega \in H_{n/2}$, $v_\alpha \neq v_\omega$ with $deg_{H(n/2)}(v_\alpha) \geq 1$ and $deg_{H(n/2)}(v_\omega) \geq 1$. See Fig. 9 (d), Figs. 10 (d)-(e), and Fig. 11.

(8) $\eta_8 = H_{n/2} \vee \overline{K_{n/2}}$, n is even, $n \geq 8$. See Fig. 12.

It is clear that $\kappa(\eta_i) \geq 4$, for $i = 1, \ldots, 8$.

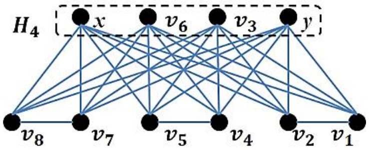


Fig. 4. $\eta_1 = H_4 \vee 3K_2$. $V(H_4) = \{v_3, v_6, x, y\}$. The three complete graphs in $3K_2$ are with $V(K_2) = \{v_i, v_{i+1}\}$, for $i = 1, 4, 7$. $deg_{\eta_1}(\varphi) \geq 6$ for $\varphi \in H_4$; and $deg_{\eta_1}(v_i) = 5$ for $i = 1, 2, 4, 5, 7, 8$.

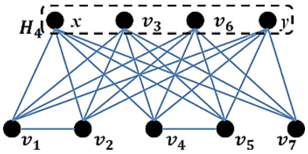
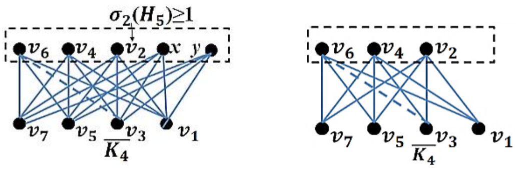


Fig. 5. $\eta_2 = H_4 \vee (2K_2 \cup K_1)$. $V(H_4) = \{v_3, v_6, x, y\}$. The two complete graphs in $2K_2$ are with $V(K_2) = \{v_i, v_{i+1}\}$, for $i = 1, 4$; and $V(K_1) = \{v_7\}$. $deg_{\eta_2}(\varphi) \geq 5$ for $\varphi \in H_4$; $deg_{\eta_2}(v_i) = 5$ for $i = 1, 2, 4, 5$ and $deg_{\eta_2}(v_7) = 4$.



(a) (b)

Fig. 6. (a) $n = 9$, $G = (H_5 \vee \overline{K}_4) - (v_3, v_6)$, with $\sigma_2(H_5) \geq 1$, $deg_G(v_3) = 4$, $deg_{H_5}(v_6) \geq 2$, $deg_G(v_6) \geq 5$, where $V(H_5) = \{v_2, v_4, v_6, x, y\}$, $V(\overline{K}_4) = \{v_1, v_3, v_5, v_7\}$; $\sigma_2(G) \geq n$, $G \in \eta_3$; (b) $n = 9$, $G'' = (H_3 \vee \overline{K}_4) - (v_3, v_6)$, $|G''| = 7$, $\sigma_2(G'') \geq n - 4$.

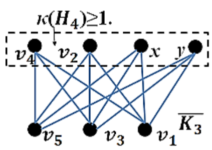
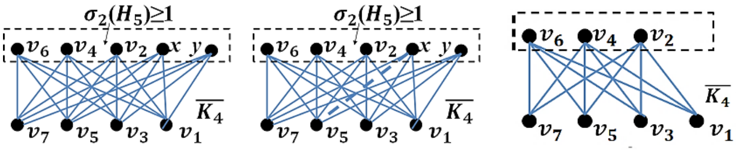


Fig. 7. $n = 7$, $\eta_4 = (H_4 \vee \overline{K}_3)$, with $\kappa(H_4) \geq 1$. $V(H_4) = \{v_2, v_4, x, y\}$, $V(\overline{K}_3) = \{v_1, v_3, v_5\}$; $deg_{\eta_4}(\varphi) > 3$ for $\varphi \in H_4$; and $deg_{\eta_4}(v_i) = 4$ for $i = 1, 3, 5$.



(a) (b) (c)

Fig. 8. (a) $n = 9$, $\eta_4 = (H_5 \vee \overline{K}_4)$, with $\sigma_2(H_5) \geq 1$, $V(H_5) = \{v_2, v_4, v_6, x, y\}$, $V(\overline{K}_4) = \{v_1, v_3, v_5, v_7\}$. $deg_{\eta_4}(\varphi) \geq 4$ for $\varphi \in H_5$; and $deg_{\eta_4}(v_i) = 5$ for $i = 1, 3, 5, 7$; (b) $n = 9$, $G = (H_5 \vee \overline{K}_4) - (x, v_5)$, with $\sigma_2(H_5) \geq 1$, $deg_G(v_5) = 4$, $deg_{H_5}(x) \geq 2$, and $deg_G(x) \geq 5$. $G \in \eta_3$; (c) $n = 9$, $G'' = (H_3 \vee \overline{K}_4)$, $|G''| = 7$, $\sigma_2(G'') \geq n - 4$.

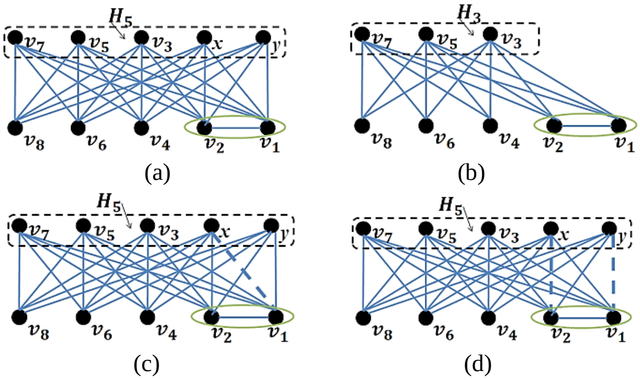


Fig. 9. (a) $n = 10$, $\eta_5 = H_5 \vee (\overline{K}_3 \cup K_2)$, $V(H_5) = \{v_3, v_5, v_7, x, y\}$, $V(\overline{K}_3) = \{v_4, v_6, v_8\}$, the complete graph in $(\overline{K}_3 \cup K_2)$ is with $V(K_2) = \{v_1, v_2\}$; $H_5 \neq \overline{K}_5$. $deg_{\eta_5}(\varphi) \geq 5$ for $\varphi \in H_5$; and $deg_{\eta_5}(v_k) = 5$ for $k = 4, 6, 8$; $deg_{\eta_5}(v_i) = 6$ for $i = 1, 2$; (b) $n = 10$, $|G''| = 8$, $G'' = H_3 \vee (\overline{K}_3 \cup K_2)$; (c) $n = 10$, $G = H_5 \vee (\overline{K}_3 \cup K_2) - (v_1, x)$ with $deg_{H_5}(x) \geq 1$. $G \in \eta_6$; (d) $n = 10$, $G = H_5 \vee (\overline{K}_4 \cup K_2) - (v_1, y) - (v_2, x)$ with $deg_{H_5}(x) \geq 1$ and $deg_{H_5}(y) \geq 1$. $G \in \eta_7$.

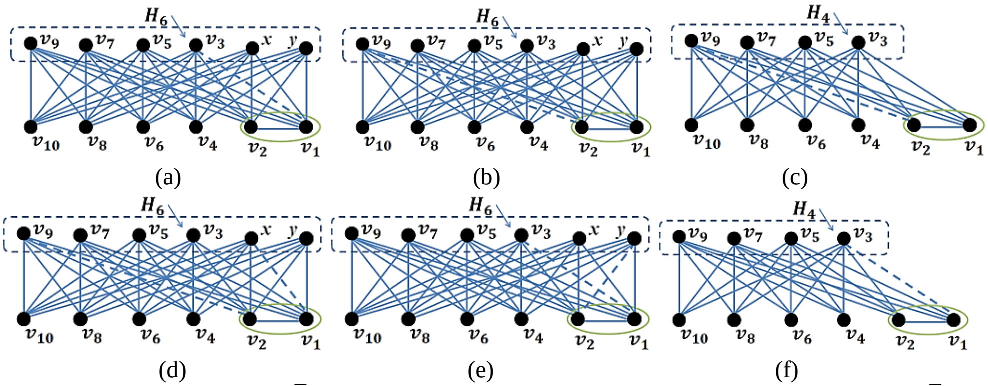
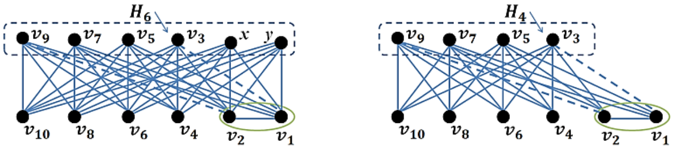


Fig. 10. (a) $n = 12$, $G = H_6 \vee (\overline{K}_4 \cup K_2) - (v_3, v_1)$ with $deg_{H_6}(v_3) \geq 1$. $G \in \eta_6$; (b) $n = 12$, $G = H_6 \vee (\overline{K}_4 \cup K_2) - (v_9, v_2)$ with $deg_{H_6}(v_9) \geq 1$. $G \in \eta_6$; (c) $n = 12$, $|G''| = 10$, $G'' = H_4 \vee (\overline{K}_4 \cup K_2) - (v_9, v_2)$; (d) $n = 12$, $G = H_6 \vee (\overline{K}_4 \cup K_2) - (v_9, v_2) - (x, v_1)$ with $deg_{H_6}(v_9) \geq 1$ and $deg_{H_6}(x) \geq 1$. $G \in \eta_6$; (e) $n = 12$, $G = H_6 \vee (\overline{K}_4 \cup K_2) - (y, v_2) - (v_3, v_1)$ with $deg_{H_6}(y) \geq 1$ and $deg_{H_6}(v_3) \geq 1$. $G \in \eta_7$; (f) $n = 12$, $|G''| = 10$, $G'' = H_4 \vee (\overline{K}_4 \cup K_2) - (v_3, v_1)$.



(a) (b)

Fig. 11. (a) $n = 12$, $\eta_7 = H_6 \vee (\overline{K}_4 \cup K_2) - (v_3, v_1) - (v_9, v_2)$ with $deg_{H_6}(v_3) \geq 1$ and $deg_{H_6}(v_9) \geq 1$; (b) $n = 12$, $|G''| = 10$, $G'' = H_4 \vee (\overline{K}_4 \cup K_2) - (v_1, v_3) - (v_9, v_2)$.

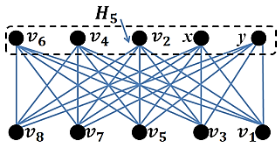


Fig. 12. $n = 10$, $\eta_8 = H_5 \vee \overline{K}_5$. $V(H_5) = \{v_2, v_4, v_6, x, y\}$, $V(\overline{K}_5) = \{v_1, v_3, v_5, v_7, v_8\}$; $deg_{\eta_8}(v_i) \geq 5$ for $i = 1\sim8$, $deg_{\eta_8}(x) \geq 5$, $deg_{\eta_8}(y) \geq 5$.

Lemma 10: If $G \in \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8\}$ with $|G| = n$, then
(1) $deg_G(u) + deg_G(v) \geq n$ holds for any nonadjacent pair of vertices $\{u, v\}$ in G;
(2) $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7$ are 1-*vertex-fault Hamiltonian*; and
(3) $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8$ are not 2-*vertex-fault Hamiltonian.*

***Proof*:**
(1) For any graph $G \in \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8\}$ with $|G| = n$, it is obvious that $deg_G(a) + deg_G(b) \geq n$ holds for any nonadjacent pair of vertices $\{a, b\}$ in G. Thus, by Ore's theorem, G is Hamiltonian.
For $\eta_4 = H_{(n+1)/2} \vee \overline{K_{(n-1)/2}}$, by $\sigma_2(H_{(n+1)/2}) \geq 1$, we can obtain that $\sigma_2(\eta_4) \geq n$. If $n = 7$, we have $\eta_4 = (H_4 \vee \overline{K}_4)$. To ensure "$\kappa(H_4 \vee \overline{K}_3) \geq 4$", we must have "$\kappa(H_4) \geq 1$".
For $\eta_3 = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}}) - (v_\alpha, v_\theta)$, $v_\alpha \in V(H_{(n+1)/2})$, and $v_\theta \in V(\overline{K_{(n-1)/2}})$, we have $deg_G(v_\theta) = (n-1)/2$. To meet the condition $deg_G(v_\alpha) + deg_G(v_\theta) \geq n$, we must have $deg_G(v_\alpha) \geq (n+1)/2$, which requires $deg_{H_{(n+1)/2}}(v_\alpha) \geq 2$.
(2) For any graph $G \in \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7\}$, it can be seen that $G \notin \{\mathcal{G}_1, \mathcal{G}_2\}$, so G is 1-*vertex-fault Hamiltonian*.
(3) For any graph G'' obtained by deleting two vertices x and y from $G \in \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8\}$, the number of components of $G'' - S$ is greater than $|S|$ for some $S \subseteq V(G'')$. Thus, by Theorem 3, G is not 2-*vertex-fault Hamiltonian*. ❑

Lemma 11: Let $G' = (V', E')$ be any graph with $|V'| = n' \geq 3$ such that for any nonadjacent vertices u and v, $deg_{G'}(u) + deg_{G'}(v) \geq n'$, and let $G = (V, E)$ with $|V| = |V' \cup \{x\}| = n' + 1 = n$. Then we have
(1) If $E = E' \cup \{(x, y) \mid y = u, v$; where $deg_{G'}(u) + deg_{G'}(v) \geq n'$, and u, v are nonadjacent in $G'\}$, then for any nonadjacent vertices u and v belonging to G', we have $deg_G(u) + deg_G(v) \geq n + 1$.
(2) If $E = E' \cup (\{(x, y) \mid y = u, v$; where $deg_{G'}(u) + deg_{G'}(v) \geq n'$, and u, v are nonadjacent in $G'\} - e)$, then for any nonadjacent vertices u and v belonging to G', we have $deg_G(u) + deg_G(v) \geq n$.

Lemma 12: $G'' = (V'', E'')$ is a graph with $|V''| = n - 2 \geq 5$. G'' is not Hamiltonian but contains a Hamiltonian path $\langle v_1, v_2, v_3, \ldots, v_{n-2}\rangle$. Let $N_{G''}(v_1) = \{v_{l_1}, v_{l_2}, v_{l_3}, \ldots, v_{l_d}\}$ with $2 = l_1 < l_2 < l_3 < \ldots < l_d$. Then $(v_{(n-2)}, v_{(l_r-1)}) \notin E''$ for each l_r with $1 \leq r \leq d$. On the other point of view, if $v_i \in N_{G''}(v_{n-2})$, then $(v_1, v_{i+1}) \notin E(G'')$ [7].

***Proof*:** If $(v_{(n-2)}, v_{(l_r-1)}) \in E''$ for some l_r with $1 \leq r \leq d$, then G'' contains a Hamiltonian cycle $\langle v_1, v_2, \ldots, v_{(l_r-1)}, v_{(n-2)}, v_{(n-3)}, \ldots, v_{l_r}, v_1\rangle$. ❑

Lemma 13: $G'' = (V'', E'')$ is a graph. If G'' is not Hamiltonian but contains a Hamiltonian path $\langle v_1, v_2, v_3, \ldots, v_{n-2}\rangle$, where $v_r, v_s \in N_{G''}(v_1)$, and $v_r, v_s \in N_{G''}(v_{n-2})$, for $r < s$, then, $(v_{r-1}, v_{s-1}) \notin E''$ and $(v_{r+1}, v_{s+1}) \notin E''$.

***Proof*:** G'' has a Hamiltonian path $\langle v_1, v_2, \ldots, v_{r-1}, v_r, v_{r+1}, \ldots, v_{s-1}, v_s, v_{s+1}, \ldots, v_{n-3}, v_{n-2}\rangle$.

If $(v_{r-1}, v_{s-1}) \in E''$, then G'' contains a Hamiltonian cycle:

$$\langle v_1, v_2, \nearrow, v_{r-1}, v_{s-1}, \searrow, v_{r+1}, v_r, v_{n-2}, \searrow, v_s, v_1\rangle.$$

If $(v_{r+1}, v_{s+1}) \in E''$, then G'' contains a Hamiltonian cycle:

$$\langle v_1, v_2, \nearrow, v_{r-1}, v_r, v_{n-2}, \searrow, v_{s+1}, v_{r+1}, \nearrow, v_{s-1}, v_s, v_1\rangle.$$ ❑

Lemma 14: $G'' = (V'', E'')$ is a graph with $|V''| = n-2 \geq 5$ that is not Hamiltonian but contains a Hamiltonian path $HP = \langle v_1, v_2, v_3, \ldots, v_{n-2}\rangle$. Let $N_{G''}(v_{n-2}) = \{v_{m_1}, v_{m_2}, v_{m_3}, \ldots, v_{m_e}\}$ with $m_1 < m_2 < m_3 < \ldots < m_e = n-3$, and $(v_1, v_{n-3}) \in E''$.
If $v_{m_r}, v_{m_s} \in N_{G''}(v_{n-2})$, w.l.o.g., $m_r < m_s$, then
(1) $(v_{m_{r-1}}, v_{m_{s-1}}) \notin E(G'')$; (2) $(v_{m_{r+1}}, v_{m_{s+1}}) \notin E(G'')$; (3) $(v_{m_{r+1}}, v_{n-2}) \notin E(G'')$; (4) $(v_{m_{r-1}}, v_{n-2}) \notin E(G'')$; (5) if $deg_{G''}(v_{n-2}) \geq (n-2)/2$, then there is a Hamiltonian cycle in G''.

***Proof*:** G'' has a Hamiltonian path $\langle v_1, v_2, \ldots, v_{m_{r-1}}, v_{m_r}, v_{m_{r+1}}, \ldots, v_{m_{s-1}}, v_{m_s}, v_{m_{s+1}}, \ldots, v_{n-3}, v_{n-2}\rangle$, and $v_{m_r}, v_{m_s} \in N_{G''}(v_{n-2})$.

(1) If $(v_{m_{r-1}}, v_{m_{s-1}}) \in E(G)$, then $\langle v_1, v_{n-3}, \searrow, v_{m_{s+1}}, v_{m_s}, v_{n-2}, v_{m_r}, v_{m_{r+1}}, \nearrow, v_{m_{s-1}}, v_{m_{r-1}}, \searrow, v_2, v_1\rangle$ is a Hamiltonian cycle.
(2) If $(v_{m_{r+1}}, v_{m_{s+1}}) \in E(G)$, then $\langle v_1, v_{n-3}, \searrow, v_{m_{s+1}}, v_{m_{r+1}}, v_{m_{r+2}}, \nearrow, v_{m_{s-1}}, v_{m_s}, v_{n-2}, v_{m_r}, \searrow, v_2, v_1\rangle$ is a Hamiltonian cycle.
(3) If $(v_{m_{r+1}}, v_{n-2}) \in E(G)$, then $\langle v_1, v_{n-3}, \searrow, v_{m_{r+1}}, v_{n-2}, v_{m_r}, \searrow, v_2, v_1\rangle$ is a Hamiltonian cycle.
(4) If $(v_{m_{r-1}}, v_{n-2}) \in E(G)$, then $\langle v_1, v_{n-3}, \searrow, v_{m_r}, v_{n-2}, v_{m_{r-1}}, \searrow, v_2, v_1\rangle$ is a Hamiltonian cycle.

Hence, in the Hamiltonian path HP, there are no two consecutive vertices that belong to $N_{G''}(v_{n-2})$. It can be seen that the vertices of $\{v_{m_1-1}, v_{m_2-1}, v_{m_3-1}, \ldots, v_{m_e-1}\} \cup \{v_{n-2}\} = N^-_{G''}(v_{n-2}) \cup \{v_{n-2}\}$ are mutually nonadjacent to each other; and the vertices of $\{v_{m_1+1}, v_{m_2+1}, v_{m_3+1}, \ldots, v_{m_e+1}\} \cup \{v_1\} = N^+_{G''}(v_{n-2}) \cup \{v_1\}$ are mutually nonadjacent to each other too.
(5) If $deg_{G''}(v_{n-2}) \geq (n-2)/2$, $|\{v_2, v_3, \ldots, v_{n-3}\}| = n-4$, and $(n-4)/2+1 = (n-2)/2 \leq deg_{G''}(v_{n-2})$, there must exist some i with $2 \leq i \leq (n-4)$ such that both v_i and v_{i+1} are adjacent to v_{n-2}. This contradicts with statements (3) and (4). ❑

Definition 15: Let $G'' = (V'', E'')$ be a graph with $|V''| = n-2$ containing a Hamiltonian path $HP = \langle v_1, v_2, v_3, \ldots, v_{n-2}\rangle$. $N_{G''}(v_1) = \{v_{l_1}, v_{l_2}, v_{l_3}, \ldots, v_{l_d}\}$ with $2 = l_1 < l_2 < l_3 < \ldots < l_d$. For the sake of simplicity, we define some notations that we will use in the rest of this paper.
(1) $S_H = \{v_i \mid (v_1, v_{i+1}) \in E(G'')\}$;

(2) $T_H = \{v_i \mid (v_i, v_{n-2}) \in E(G'')\}$, which is $N_{G''}(v_{n-2})$;
(3) $W_H = \{v_i \mid v_i \in V(G'') - \{v_{n-2}\}, v_i \notin S_H$, and $v_i \notin T_H\}$;
(4) $N^-_{G''}(v_{n-2}) = \{v_{s-1} \mid v_s \in N_{G''}(v_{n-2})\}$;
(5) $N^+_{G''}(v_{n-2}) = \{v_{s+1} \mid v_s \in N_{G''}(v_{n-2})\}$;
(6) $VLD = \{v_i \mid l_d + 1 \le i \le n-2\}$;
(7) $G[VLD \cup \{v_{l_d}\}]$ denote the subgraph induced by $VLD \cup \{v_{l_d}\}$;
(8) K_{VLD} be a complete graph formed by the vertices of VLD;
(9) K'_{VLD} denote a graph removing some (none, one, or more) *vertex-disjoint-edges* of K_{VLD};
(10) $K_{VLD\cup\{v_{l_d}\}}$ denote the complete graph with vertex set $VLD \cup \{v_{l_d}\}$;
(11) $K'_{VLD\cup\{v_{l_d}\}}$ denote a graph obtained by removing some (none, one, or more) *vertex-disjoint-edges* from the complete graph $K_{VLD\cup\{v_{l_d}\}}$;
(12) $V_{in} = \{v_i \mid v_i \notin N_{G''}(v_1), i < l_d\}$;
(13) $V_d = \{v_1\} \cup \{v_i \mid v_i \in N_{G''}(v_1), i < l_d\}$;
(14) $G[V_d \cup \{v_{l_d}\}]$ denote the subgraph induced by $V_d \cup \{v_{l_d}\}$;
(15) $K_{V_d\cup\{v_{l_d}\}}$ be a complete graph formed by the vertices of $V_d \cup \{v_{l_d}\}$;
(16) $K'_{V_d\cup\{v_{l_d}\}}$ denote a graph removing some (none, one, or more) *vertex-disjoint-edges* of $K_{V_d\cup\{v_{l_d}\}}$.

Table 1. Hamiltonian path and row of "S_H, T_H, W_H".

	1	2	...	t	...	w	...	$b-1$	$b = l_d$	...	$n-3$	$n-2$
1	v_1	v_2	...	v_t	…	v_w	...	v_{b-1}	v_b	…	v_{n-3}	v_{n-2}
2		v_{l_1}					...	…	v_{l_d}			
3	S_H	…	…	T_H	…	W_H	...	S_H		...	T_H	

In Table 1, we put each element of the Hamiltonian path HP in the first row and examine every element. There is no doubt that $v_1 \in S_H$ because $v_2 = v_{l_1}$. So, we denote the (3,1) entry as S_H. For element v_2, we denote the (3,2) entry as S_H if $v_2 \in S_H$, as T_H if $v_2 \in T_H$, or as W_H if $v_2 \in W_H$. We repeatedly perform the examination from v_3 to v_{n-4}, and then place the appropriate symbol on the corresponding entry. Since $(v_{n-3}, v_{n-2}) \in E(G'')$ indicates $v_{n-3} \in T_H$, we denote the (3, n–3) entry as T_H. When $v_b = v_{l_d}$, the (3, b–1) entry is denoted by S_H. Since l_d is the largest index, it can be seen that either $v_i \in T_H$ or $v_i \in W_H$ for each i where $b \le i \le n-4$. Furthermore, T_H is the entry of (3, t) which implies $(v_t, v_{n-2}) \in E(G'')$. When W_H is the entry of (3, w), then we must have $(v_w, v_{n-2}) \notin E(G'')$ and $(v_1, v_{w+1}) \notin E(G'')$.

Lemma 16: Let $G = (V, E)$ be a graph with $|V| = n \ge 7$, and $\sigma_2(G) \ge n$. For some vertices $x, y \in V$, let $G'' = (V'', E'')$ with $V'' = V - \{x, y\}$ and $E'' = E - \{(x, s)$ and $(y, t) \mid s, t \in V\}$. Suppose that G'' is not Hamiltonian but contains a Hamiltonian path $\langle v_1, v_2, v_3, \ldots, v_{n-2}\rangle$. Then the following three statements are true:

(1) $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$ or $n-3$.
(2) If $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-3$, then $N_{G''}(v_{n-2}) = \{v_a \mid 1 \le a \le n-3\} - \{v_b \mid (v_1, v_{b+1}) \in E(G'')\}$; Namely, if any vertex v_a in $G''-\{v_1\}$ is not adjacent to v_1, then v_{a-1} must be adjacent to v_{n-2}.
(3) If $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$, then $N_{G''}(v_{n-2}) \subset \{v_a \mid 1 \le a \le n-3\} - \{v_b \mid (v_1, v_{b+1}) \in E(G)\}$, and $|N_{G''}(v_{n-2})| = |\{v_a \mid 1 \le a \le n-3\}| - |\{v_b \mid (v_1, v_{b+1}) \in E(G)\}| - 1$ [7].

***Proof*:** Note that S_H, T_H, W_H are defined in Definition 15, and $deg_{G''}(u)+deg_{G''}(v) \geq n-4$ holds for any nonadjacent pair $\{u, v\}\subset V''$. If G'' is not Hamiltonian but contains a Hamiltonian path $\langle v_1, v_2, v_3, \ldots, v_{n-2}\rangle$, then it is easy to see that $(v_1, v_{n-2})\notin E''$, which implies $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) \geq n-4$.

Since $v_{n-2}\notin S_H\cup T_H$, we have $|S_H\cup T_H| < n-2$. Furthermore, we claim that $S_H\cap T_H = \emptyset$. If $S_H\cap T_H \neq \emptyset$, there must be some vertex, called v_α, such that $v_\alpha\in S_H\cap T_H$. This implies that $(v_1, v_{\alpha+1})\in E(G'')$, and $(v_\alpha, v_{n-2})\in E(G'')$. By Lemma 12, this is a contradiction.

Clearly, $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = |S_H\cup T_H| < n-2$. Consequently, we have $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-3$ or $n-4$ and $T_H\subseteq V(G'')-(S_H\cup\{v_{n-2}\})$. Note that $|S_H\cup T_H| + |\{v_{n-2}\}| + |W_H| = n-2$. If $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-3$, then $|W_H| = 0$ and $N_{G''}(v_{n-2}) = \{v_a|1 \leq a \leq n-3\} - \{v_b|(v_1, v_{b+1})\in E(G)\}$. If $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$, then $|W_H| = 1$ and $|N_{G''}(v_{n-2})| = |\{v_a|1 \leq a \leq n-3\}| - |\{v_b|(v_1, v_{b+1})\in E(G)\}| - 1$. ❑

The above proof shows that when $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-3$, we must have either $v_i\in S_H$ or $v_i\in T_H\ \forall\ v_i\in V''-\{v_{n-2}\}$, which indicates that there is a row of "S_H, T_H" corresponding to the Hamiltonian path. On the other hand, when $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$, then we must have any one of the three: (1) $v_i\in S_H$; (2) $v_i\in T_H$; (3) $v_i\in W_H\ \forall\ v_i\in V''-\{v_{n-2}\}$, which indicates that there is a row of "S_H, T_H, W_H" corresponding to the Hamiltonian path. In the following, we will only use "S_H, T_H, W_H" to represent a row of "S_H, T_H, W_H" or a row of "S_H, T_H".

Lemma 17: Let $G'' = (V'', E'')$ be a graph with $|V''| = n-2 \geq 7$, and $\sigma_2(G'') \geq n-4$. Suppose that G'' is not Hamiltonian but contains a Hamiltonian path $\langle v_1, v_2, v_3, \ldots, v_{n-3}, v_{n-2}\rangle$ with $N_{G''}(v_{n-2}) = \{v_{m_1}, v_{m_2}, v_{m_3}, \ldots, v_{m_e}\}$, in which $m_1 < m_2 < m_3 < \ldots < m_e = n-3$, and $(v_1, v_{n-3})\in E''$. Then, we have

(1) $(n-5)/2 \leq deg_{G''}(v_{n-2}) \leq (n-3)/2$.

(2) If n is odd, for each element $v_{m_{t-1}}\in N^-_{G''}(v_{n-2})$, we have $(n-5)/2 \leq deg_{G''}(v_{m_{t-1}}) \leq (n-3)/2$; and for each element $v_{m_{t+1}}\in N^+_{G''}(v_{n-2})$, we have $(n-5)/2 \leq deg_{G''}(v_{m_{t+1}}) \leq (n-3)/2$.

(3) If n is even, for each element $v_{m_{t-1}}\in N^-_{G''}(v_{n-2})$, we have $(n-4)/2 \leq deg_{G''}(v_{m_{t-1}}) \leq (n-2)/2$; and for each element $v_{m_{t+1}}\in N^+_{G''}(v_{n-2})$, we have $(n-4)/2 \leq deg_{G''}(v_{m_{t+1}}) \leq (n-2)/2$.

Note that if there exists one $v_{m_{t-1}}$ with $deg_{G''}(v_{m_{t-1}}) = (n-2)/2$, then, $v_{m_{t-1}}$ is the only one vertex of degree $(n-2)/2$ in $N^+_{G''}(v_{n-2})$. Similarly, if there exists one $v_{m_{t+1}}$ with $deg_{G''}(v_{m_{t+1}}) = (n-2)/2$, then, $v_{m_{t+1}}$ is the only one vertex of degree $(n-2)/2$ in $N^+_{G''}(v_{n-2})$.

(4) $deg_{G''}(v_{m_{t-1}}) + deg_{G''}(v_{m_{s-1}}) \leq (n-3)$ for each $v_{m_{t-1}}, v_{m_{s-1}}\in N^-_{G''}(v_{n-2})$; and $deg_{G''}(v_{m_{t+1}}) + deg_{G''}(v_{m_{s+1}}) \leq (n-3)$ for each $v_{m_{t+1}}, v_{m_{s+1}}\in N^+_{G''}(v_{n-2})$.

(5) If $deg_{G''}(v_{n-2}) = (n-5)/2$, then $deg_{G''}(v_{m_{t-1}}) = (n-3)/2$, $\forall v_{m_{t-1}}\in N^-_{G''}(v_{n-2})$; and $deg_{G''}(v_{m_{t+1}}) = (n-3)/2$, $\forall v_{m_{t+1}}\in N^+_{G''}(v_{n-2}) - \{v_{n-2}\}$.

(6) If there is an element $v_{m_{t-1}}\in N^-_{G''}(v_{n-2})$, with degree $(n-5)/2$, then the degrees of all other vertices $v_{m_{s-1}}\in N^-_{G''}(v_{n-2})$ are equal to $(n-3)/2$, and $deg_{G''}(v_{n-2}) = (n-3)/2$.

(7) If there is an element $v_{m_{t+1}}\in N^+_{G''}(v_{n-2})$ with degree $(n-5)/2$, then the degrees of all other vertices, $v_{m_{s+1}}\in N^+_{G''}(v_{n-2})$, are equal to $(n-3)/2$.

***Proof*:**

(1) Lemma 14 (5) has proved that $deg_{G''}(v_{n-2}) \leq (n-3)/2$ for n is odd; and $deg_{G''}(v_{n-2}) \leq (n-4)/2$ for n is even. By Lemma 14, "the vertices of $N^-_{G''}(v_{n-2})\cup\{v_{n-2}\}$ are mutually non-

adjacent to each other", we have $deg_{G''}(v_{m_t\text{-}1}) + deg_{G''}(v_{n\text{-}2}) \geq n - 4$, for every $v_{m_t\text{-}1} \in N^-_{G''}(v_{n\text{-}2})$. If $deg_{G''}(v_{n\text{-}2}) < (n\text{–}5)/2$, then we must have $deg_{G''}(v_{m_t\text{-}1}) > (n\text{–}3)/2$, that is $deg_{G''}(v_{m_t\text{-}1}) \geq (n\text{–}1)/2$. This implies that $deg_{G''}(v_{m_s\text{-}1}) + deg_{G''}(v_{m_t\text{-}1}) \geq (n\text{–}2)$, for every two vertices $v_{m_s\text{-}1}, v_{m_t\text{-}1} \in N^-_{G''}(v_{n\text{-}2})$. By Lemma 14, $G''+(v_{m_s\text{-}1}, v_{m_t\text{-}1})$ is Hamiltonian, but, by Theorem 2, G'' is Hamiltonian, which is a contradiction to the assumption that G'' is not Hamiltonian.

(2) When n is odd, for each $v_{m_t\text{-}1} \in N^-_{G''}(v_{n\text{-}2})$, if $deg_{G''}(v_{m_t\text{-}1}) < (n\text{–}5)/2$, then, by (1), $deg_{G''}(v_{m_t\text{-}1}) + deg_{G''}(v_{n\text{-}2}) < n\text{–}4$, which is a contradiction. On the other hand, since the vertices of $N^-_{G''}(v_{n\text{-}2}) \cup \{v_{n\text{-}2}\}$ are mutually nonadjacent to each other, for each $v_{m_t\text{-}1} \in N^-_{G''}(v_{n\text{-}2})$, we have $N_{G''}(v_{m_t\text{-}1}) \subseteq \{v_1, v_2, v_3, \ldots, v_{n\text{-}3}, v_{n\text{-}2}\} - \{\{v_{m_1\text{-}1}, v_{m_2\text{-}1}, v_{m_3\text{-}1}, \ldots, v_{m_e\text{-}1}\} \cup \{v_{n\text{-}2}\}\}$, hence

$$deg_{G''}(v_{m_t\text{-}1}) \leq (n\text{–}2) - |N^-_{G''}(v_{n\text{-}2}) \cup \{v_{n\text{-}2}\}| \leq (n\text{–}3) - |N_{G''}(v_{n\text{-}2})|. \tag{1}$$

By statement (1) of this Lemma, when n is odd, we have $deg_{G''}(v_{n\text{-}2}) = (n\text{–}3)/2$ or $deg_{G''}(v_{n\text{-}2}) = (n\text{–}5)/2$. If $deg_{G''}(v_{n\text{-}2}) = (n\text{–}3)/2$, Eq. (1) shows that $deg_{G''}(v_{m_t\text{-}1}) \leq (n\text{–}3) - (n\text{–}3)/2 \leq (n\text{–}3)/2$. If $deg_{G''}(v_{n\text{-}2}) = (n\text{–}5)/2$, we have $deg_{G''}(v_{m_t\text{-}1}) \leq (n\text{–}1)/2$ by Eq. (1). In order to maintain the condition of $deg_{G''}(v_{m_t\text{-}1}) + deg_{G''}(v_{n\text{-}2}) \geq n\text{–}4$, for each $v_{m_t\text{-}1} \in N^-_{G''}(v_{n\text{-}2})$, we must have

$$deg_{G''}(v_{m_t\text{-}1}) \geq n - 4 - deg_{G''}(v_{n\text{-}2}) \geq (n\text{–}3)/2. \tag{2}$$

Hence, if there exists a vertex $v_{m_s\text{-}1} \in N^-_{G''}(v_{n\text{-}2})$ and $deg_{G''}(v_{m_s\text{-}1}) = (n\text{–}1)/2$, then, by Eq. (2), we will have $deg_{G''}(v_{m_t\text{-}1}) + deg_{G''}(v_{m_s\text{-}1}) \geq n\text{–}2$. Thus, by Lemma 14, $G''+(v_{m_s\text{-}1}, v_{m_t\text{-}1})$ is Hamiltonian, and by Lemma 2, G'' is Hamiltonian, which is a contradiction. Therefore, we obtain $(n\text{–}5)/2 \leq deg_{G''}(v_{m_t\text{-}1}) \leq (n\text{–}3)/2$, for each $v_{m_t\text{-}1} \in N^-_{G''}(v_{n\text{-}2})$.

In a similar manner, for each $v_{m_t+1} \in N^+_{G''}(v_{n\text{-}2})$, we have $(n\text{–}5)/2 \leq deg_{G''}(v_{m_t+1}) \leq (n\text{–}3)/2$.

(3) When n is even, in a similar manner, we can prove that the statement holds.

(4) Obviously, the statement is true.

(5) Since $deg_{G''}(v_{n\text{-}2}) + deg_{G''}(v_{m_t\text{-}1}) \geq (n\text{–}4)$, $\forall v_{m_t\text{-}1} \in N^-_{G''}(v_{n\text{-}2})$, if $deg_{G''}(v_{n\text{-}2}) = (n\text{–}5)/2$, we have $deg_{G''}(v_{m_t\text{-}1}) \geq (n\text{–}3)/2$. However, we have learned from (2) that $deg_{G''}(v_{m_t\text{-}1}) \leq (n\text{–}3)/2$. Hence, we have $deg_{G''}(v_{m_t\text{-}1}) = (n\text{–}3)/2$.

Similarly, we can prove that $deg_{G''}(v_{m_t+1}) = (n\text{–}3)/2$, $\forall v_{m_t+1} \in N^+_{G''}(v_{n\text{-}2}) - \{v_{n\text{-}2}\}$.

The proofs of (6) and (7) are similar to the proof of (5). This completes the proof. ❑

Theorem 18: (Erdős) Suppose that G is a graph such that any two nonadjacent vertices of G satisfying $deg_G(u)+deg_G(v) \geq n(G)+1$. Then G is Hamiltonian-connected [1, 10, 11].

Lemma 19: $G'' = (V'', E'')$ is not Hamiltonian but contains a hamiltonian path $HP = \langle v_1, v_2, v_3, \ldots, v_{n\text{-}2}\rangle$ with $|V''| = n - 2 \geq 7$, $\sigma_2(G'') \geq n - 4$, $N_{G''}(v_1) = \{v_{l_1}, v_{l_2}, v_{l_3}, \ldots, v_{l_d}\}$ with $2 = l_1 < l_2 < l_3 < \ldots < l_d$. Then the following statements are true.

(1) If $deg_{G''}(v_1)+deg_{G''}(v_{n\text{-}2}) = n - 3$, then $deg_{G''}(v_i)+deg_{G''}(v_1) = n - 3$ or $n - 4$ for any i, where $l_d + 1 \leq i \leq n - 2$.

(2) If $l_d \leq n - 6$, then $G[VLD \cup \{v_{l_d}\}]$ is Hamiltonian-connected.

(3) If $|V_{in}| = 0$, then
  (i) G'' is not 2-connected;
  (ii) $G'' \in K'_{Vd} : v_{l_d} : K'_{VLD}$.

Note that VLD, $G[VLD \cup \{v_{l_d}\}]$, $K_{VLD\cup\{v_{l_d}\}}$, $K'_{VLD\cup\{v_{l_d}\}}$, V_{in}, V_d, $G[V_d \cup \{v_{l_d}\}]$, $K_{V_d\cup\{v_{l_d}\}}$, $K'_{V_d\cup\{v_{l_d}\}}$, K_{VLD} and K'_{VLD} are defined in Definition 15.

***Proof*:** Let $l_d = b$. In Table 2, we place the vertices of the Hamiltonian path $HP = \langle v_1, \ldots,$

$v_{l_d\text{-}1}$, v_{l_d}, v_{l_d+1}, …, v_{l_d+i}, v_{l_d+i+1}, v_{l_d+i+2}, …, v_{l_d+k}, …, $v_{n\text{-}3}$, $v_{n\text{-}2}\rangle$ on the entries in the first row, in which v_{l_d} is in the b^{th} column, v_{l_d+i} is in the $(l_d+i)^{\text{th}}$ column, v_{l_d+k} is in the $(l_d+k)^{\text{th}}$ column, and so on, where $i < k$. The entries of the 2^{nd} and 3^{rd} rows are assigned to be "S_H, T_H, W_H" corresponding to the Hamiltonian path *HP* (note that S_H, T_H, and W_H are defined in Definition 15).

Table 2. Hamiltonian paths *HP* and P_{l_d+i}.

	1	.	***b*–1**	*b*	l_d+1	.	$\boldsymbol{l_d+i}$	$\boldsymbol{l_d+i+1}$	$\boldsymbol{l_d+i+2}$	.	$\boldsymbol{n+i\text{-}k\text{-}2}$	.	$\boldsymbol{l_d+k}$	.	***n*–3**	***n*–2**
1	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	.	v_{l_d+i}	v_{l_d+i+1}	v_{l_d+i+2}	.	$v_{n+i\text{-}k\text{-}2}$	.	v_{l_d+k}	.	$v_{n\text{-}3}$	$v_{n\text{-}2}$
2	S_H		S_H	T_H	T_H		T_H	T_H	…		T_H	T_H	T_H		T_H	
3	S_H		S_H	T_H	T_H		W_H	T_H	…		…	T_H	T_H		T_H	
4	…		…	…	…		…	…	…		…				…	…
5	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	$v_{n\text{-}2}$	.	$v_{n\text{-}i\text{-}1}$	$v_{n\text{-}i\text{-}2}$	$v_{n\text{-}i\text{-}3}$	.	$v_{k\text{-}i+1+l_d}$	.	$v_{n\text{-}k\text{-}1}$	.	v_{l_d+2}	v_{l_d+1}
6	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	.	$v_{n\text{-}i}$	$v_{n\text{-}i\text{-}1}$	$v_{n\text{-}i\text{-}2}$	.	$v_{k\text{-}i+2+l_d}$	.	$v_{n\text{-}k}$	.	v_{l_d+3}	v_{l_d+2}
7	**…**		**…**	**…**	**…**		**…**	**…**	**…**		**…**				**…**	**…**
8	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	.	$v_{n\text{-}2}$	$v_{n\text{-}3}$	**…**	.	v_{l_d+k}	.	$v_{n+i\text{-}k\text{-}2}$	.	v_{l_d+i+1}	v_{l_d+i}
9	S_H		S_H	T_H	T_H		T_H	T_H	…		W_H		T_H		T_H	
10	S_H		S_H	T_H	T_H		W_H	T_H	…		T_H		T_H		T_H	
11	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	.	v_{l_d+i}	$v_{n\text{-}2}$	$v_{n\text{-}3}$	.	v_{l_d+k+1}	.	$v_{n+i\text{-}k\text{-}1}$	.	v_{l_d+i+2}	v_{l_d+i+1}
12	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	.	v_{l_d+i}	v_{l_d+i+1}	$v_{n\text{-}2}$	.	v_{l_d+k+2}	.	$v_{n+i\text{-}k}$	.	v_{l_d+i+3}	v_{l_d+i+2}
13	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	.	v_{l_d+i}	v_{l_d+i+2}	v_{l_d+i+3}	.	$v_{n+i\text{-}k\text{-}1}$	.	v_{l_d+k+1}	.	$v_{n\text{-}2}$	v_{l_d+i+1}
14	**…**	.	**…**	**…**	**…**	.	**…**	**…**	**…**	.	**…**	.		.	**…**	**…**
15	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	.	v_{l_d+i}	v_{l_d+i+1}	v_{l_d+i+2}	.	$v_{n+i\text{-}k\text{-}2}$	.	$v_{n\text{-}2}$	.	v_{l_d+k+1}	v_{l_d+k}
16	S_H		S_H	T_H	T_H		W_H	T_H	…		T_H		T_H		T_H	…
17	**…**	.	**…**	**…**	**…**	.	**…**	**…**	**…**	.	**…**	.		.	**…**	**…**
18	v_1	.	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	.	v_{l_d+i}	v_{l_d+i+1}	v_{l_d+i+2}	.	$v_{n+i\text{-}k\text{-}2}$	.	v_{l_d+k}	.	$v_{n\text{-}2}$	$v_{n\text{-}3}$

(1) If $deg_{G''}(v_1) + deg_{G''}(v_{n\text{-}2}) = n–3$, then, by Lemma 16, we have $(v_x, v_{n\text{-}2}) \in E''$, and the entries of (2, x) are all T_H for each x, where $l_d \leq x \leq n–3$. This results in the following new Hamiltonian paths:

$P_{l_d+1} = \langle v_1, v_2, \ldots, v_{l_d}, v_{n\text{-}2}, v_{n\text{-}3}, \ldots, v_{l_d+1}\rangle$,
$P_{l_d+2}, \ldots, P_{l_d+i} = \langle v_1, v_2, \ldots, v_{l_d}, v_{l_d+1}, v_{l_d+2}, \ldots, v_{l_d+i\text{-}1}, v_{n\text{-}2}, v_{n\text{-}3}, \ldots, v_{l_d+i}\rangle$,
$P_{l_d+i+1}, P_{l_d+i+2}, \ldots, P_{l_d+k}, \ldots, P_{n\text{-}3} = \langle v_1, v_2, \ldots, v_{l_d}, v_{l_d+1}, v_{l_d+2}, v_{l_d+3}, \ldots, v_{n\text{-}4}, v_{n\text{-}2}, v_{n\text{-}3}\rangle$.

They are illustrated in the 5^{th}, 6^{th}, 8^{th}, 11^{th}, 12^{th}, 15^{th}, and 18^{th} rows of Table 2. We can see that v_{l_d+k} in the Hamiltonian path P_{l_d+i} is located in the $(n+i–k–2)^{\text{th}}$ column, not in the $(l_d+k)^{\text{th}}$ column.

In the 5^{th} row of Table 2, since v_1 and v_{l_d+1} are the two end vertices of the Hamiltonian path P_{l_d+1}, we have, by Lemma 16, $deg_{G''}(v_{l_d+1}) + deg_{G''}(v_1) = n–3$ or $n–4$. This can be applied to the rest rows. Thus, we have $deg_{G''}(v_i) + deg_{G''}(v_1) = n–3$ or $n–4$, for any i, where $l_d+1 \leq i \leq n–3$.

For each Hamiltonian path P_{l_d+i}, where $1 \leq i \leq n - l_d - 3$, the entries may vary starting from the column (l_d+1) to column $(n–2)$, but the entries before the column (l_d+1) remain unchanged. In other words, we can find that the vertices v_1, v_2, …, v_{l_d} are always in the same positions respectively, as observed in a comparison of the Hamiltonian path *HP* and all Hamiltonian paths P_{l_d+i}, where $1 \leq i \leq n - l_d - 3$. So, all vertices belonging to $N_{G''}(v_1)$

remain in the same entries, respectively. This indicates that each S_H in the row of "S_H, T_H, W_H" corresponding to each Hamiltonian path must remain in the same positions, respectively. Moreover, besides S_H, all other entries are T_H only, or several T_H and one W_H.

(2) We consider the following cases:

Case 1 $deg_{G''}(v_1)+deg_{G''}(v_{n-2})=n–3$.

As proved previously, we have $(v_x, v_{n-2})\in E''$ for $l_d \le x \le n–3$ and Hamiltonian paths P_{l_d+i} for $1\le i \le n–l_d–3$.

Case 1.1 If $deg_{G''}(v_1) + deg_{G''}(v_{l_d+i}) = n–3$ for each i, where $1\le i \le n–l_d–3$, we have, by Lemma 16, $(v_x, v_{l_d+i})\in E''$, for $l_d \le x \le n–3$, $x \ne l_d+i$. So, the induced graph $G[VLD \cup \{v_{l_d}\}]$ is a complete graph, denoted by $K_{VLD\cup\{v_{l_d}\}}$. Consequently, $G[VLD \cup \{v_{l_d}\}]$ is Hamiltonian connected.

Case 1.2 If there is an i, where $1\le i \le n–l_d–3$, such that $deg_{G''}(v_1) + deg_{G''}(v_{l_d+i}) = n–4$, then, by Lemma 16, there exists a vertex v_α such that $(v_\alpha, v_{l_d+i})\notin E''$, and $(v_{\alpha+1}, v_1)\notin E(G'')$. By Definition 15, we have $v_\alpha\in W_H$. We can find that only one entry is different in a comparison of the row of "S_H, T_H, W_H" that corresponds to the Hamiltonian path P_{l_d+i} and the row of "S_H, T_H, W_H" that corresponds to the Hamiltonian path HP. Among the entries formed by S_H and T_H, which are corresponding to the Hamiltonian path HP, only one T_H is replaced by W_H in the row of "S_H, T_H, W_H" corresponding to P_{l_d+i}. The entries of P_{l_d+i} are shown in the 8th row of Table 2.

Case 1.2.1 $\alpha < l_d$.

This indicates that the W_H is located before the $(l_d)^{\text{th}}$ column, and the vertex v_{l_d+i} connects to each v_x, where $l_d \le x \le n–2$, $x \ne l_d+i$.

Case 1.2.2 $\alpha \ge l_d$.

This indicates that the W_H is located in or after column l_d. W.L.O.G., suppose $\alpha = l_d+k$, $k > i$. Then we have $(v_{l_d+k}, v_{l_d+i}) \notin E''$. It can be seen that in the Hamiltonian path P_{l_d+i}, v_{l_d+k} is in the $(n+i–k–2)^{\text{th}}$ column; hence in the row of "S_H, T_H, W_H" corresponding to P_{l_d+i}, W_H is in the $(n+i–k–2)^{\text{th}}$ column. See the 9th row of Table 2. By Lemma 16, there is only one W_H in the row of "S_H, T_H, W_H" corresponding to Hamiltonian path. We can see that whenever $(v_{l_d+k}, v_{l_d+i})\notin E''$ in the corresponding Hamiltonian path P_{l_d+i}, we will always have $(v_{l_d+i}, v_{l_d+k})\notin E''$ in the corresponding Hamiltonian path P_{l_d+k}, where l_d+i and l_d+k are not consecutive integers. The entries of P_{l_d+k} are shown in the 15th row of Table 2. In addition, v_{l_d+i} is in the $(l_d+i)^{\text{th}}$ column in the Hamiltonian path P_{l_d+k}; hence, in the row of "S_H, T_H, W_H" corresponding to P_{l_d+k}, the only one "W_H" must be in the $(l_d+i)^{\text{th}}$ column. See the 16th row of Table 2.

In the graph $G[VLD \cup \{v_{l_d}\}]$, if $deg_{G''}(v_1) + deg_{G''}(v_{l_d+s}) = n–3$ for all other Hamiltonian paths P_{l_d+s}, where $1\le s \le n–l_d–3$, $s \ne i$, and $s \ne k$, then $G[VLD \cup \{v_{l_d}\}]$ is a graph removing one edge (v_{l_d+i}, v_{l_d+k}) from the complete graph $K_{VLD\cup\{v_{l_d}\}}$ (Definition 15 (10)), where l_d+i and l_d+k are not consecutive integers. Thus $G[VLD\cup\{v_{l_d}\}]$ is Hamiltonian-connected. On the other hand, there are total $(n–2)–l_d$ Hamiltonian paths, so we have $(n–2)–l_d$ rows of "S_H, T_H, W_H", which indicates that the number of "W_H" is at most $(n–2)–l_d$. Therefore, $G[VLD \cup \{v_{l_d}\}]$ is the graph $K'_{VLD\cup\{v_{l_d}\}}$ obtained by removing at most $[(n–2)–l_d]/2$ *vertex-disjoint-edges* from the complete graph $K_{VLD\cup\{v_{l_d}\}}$, where $|V(K_{VLD\cup\{v_{l_d}\}})| = (n–1) – l_d$. The degree of each vertex of $G[VLD \cup \{v_{l_d}\}]$ is one less than or equal to $|VLD \cup \{v_{l_d}\}| – 1$. Because $l_d \le n–6$ means $|G[VLD \cup \{v_{l_d}\}]| \ge 5$, we prove that $G[VLD \cup \{v_{l_d}\}]$ is Hamiltonian-connected by Theorem 18.

Case 2 $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$.

By Lemma 16, there exists a vertex v_α, such that $(v_\alpha, v_{n-2}) \notin E''$, and $(v_1, v_{\alpha+1}) \notin E(G'')$; by Definition 15, $v_\alpha \in W_H$, and there is only one "W_H" in the row of entries formed by "S_H, T_H, W_H" corresponding to the Hamiltonian path *HP*.

Case 2.1 $\alpha < l_d$

In this situation, the W_H is located before the $(l_d)^{th}$ column, so the vertex v_{n-2} connects to each v_x for $l_d \leq x \leq n-3$, which shows that P_{l_d+i} are Hamiltonian paths for $1 \leq i \leq n-l_d-3$. Further analysis is similar to Case 1.

Case 2.2 $\alpha \geq l_d$.

In this situation, the W_H is located in or after column l_d. Suppose v_{l_d+i} is not a neighbor of v_{n-2}, that is $(v_{l_d+i}, v_{n-2}) \notin E''$. Then, we will always have $(v_{n-2}, v_{l_d+i}) \notin E''$ in the corresponding Hamiltonian path P_{l_d+i}. The entries of *HP* and of P_{l_d+i} are shown in the 1st row and the 8th row of Table 2 respectively, which show that the "W_H"s in rows of "S_H, T_H, W_H" that correspond to the Hamiltonian path *HP* and Hamiltonian path P_{l_d+i}, are both in the column (l_d+i), as illustrated in the 3rd row and the 10th row of Table 2.

By Lemma 16, we have $(v_x, v_{n-2}) \in E''$, and the entries of $(3, x)$ are all T_H for each x, where $l_d \leq x \leq n-3$, $x \neq l_d+i$. This leads to new Hamiltonian paths, $P_{l_d+1}, \ldots, P_{l_d+i}, P'_{l_d+i+1}, P_{l_d+i+2}, \ldots, P_{n-3}$, which are similar to statement (1) but its P_{l_d+i+1} is replaced by
$P'_{l_d+i+1} = \langle v_1, v_2, \ldots, v_{l_d}, v_{l_d+1}, \ldots, v_{l_d+i}, v_{l_d+i+2}, v_{l_d+i+3}, \ldots, v_{n-3}, v_{n-2}, v_{l_d+i+1}\rangle$.
The entries of P'_{l_d+i+1} are shown in the 13th row of Table 2.

The reason to replace P_{l_d+i+1} by P'_{l_d+i+1} is given below.
Because $(v_{l_d+i}, v_{n-2}) \notin E''$, P_{l_d+i+1} does not exist. But P_{l_d+i+2} exists as shown in the 12th row of Table 2. If $deg_{G''}(v_1)+deg_{G''}(v_{l_d+i+2}) = n-3$, then $(v_x, v_{l_d+i+2}) \in E''$, for $l_d \leq x \leq n-3$ and $x \neq l_d+i+2$. If $deg_{G''}(v_1) + deg_{G''}(v_{l_d+i+2}) = n-4$, then there is one and only one v_α with $(v_\alpha, v_{l_d+i+2}) \notin E''$ and $v_\alpha \in W_H$ in the row of "S_H, T_H, W_H" corresponding to the Hamiltonian path P_{l_d+i+2}. If $(v_{l_d+i}, v_{n-2}) \notin E''$ and $(v_{l_d+i}, v_{l_d+i+2}) \notin E''$, then there are two vertices not connecting to v_{l_d+i}. This is a contradiction. Hence the only one v_α must not be v_{l_d+i}, indicating $(v_{l_d+i}, v_{l_d+i+2}) \in E''$. Through a proper conversion, we can obtain P'_{l_d+i+1} from P_{l_d+i+2}. Further analysis is similar to Case 1. In addition, from this statement, we can also obtain that there exists a Hamiltonian path from v_1 to any vertex in *VLD*, and $deg_{G''}(v_i) + deg_{G''}(v_1) = n-3$ or $n-4$ for any i, where $l_d+1 \leq i \leq n-2$.

(3) $|V_{in}| = 0$

(i) When $|V_{in}| = 0$, the Hamiltonian path can be written as $\langle v_1, v_2 = v_{l_1}, v_3 = v_{l_2}, \ldots, v_{d+1} = v_{l_d}, \ldots, v_{n-2}\rangle$. By the proof of (2), none of $\{v_{l_d+1}, v_{l_d+2}, \ldots, v_{n-2}\}$ is adjacent to $\{v_1, v_2, \ldots, v_{l_d-1}\}$. Obviously, $\{v_{l_d}\}$ is a one element vertex cut. Therefore, graph G'' is not 2-connected.

(ii) There are two cases to consider.

Case 1. If $deg_{G''}(v_1)+deg_{G''}(v_{n-2}) = n-3$, then $deg_{G''}(v_{n-2}) = (n-3)-d$.
In Case 1 of (2), we have proved that $G[VLD \cup \{v_{l_d}\}]$ is a graph removing some (none, one, or more) vertex-disjoint edges of the complete graph $K_{VLD\cup\{v_{l_d}\}}$.

Since $deg_{G''}(v_{n-2})+deg_{G''}(v_i) \geq n-4$ holds for any $1 < i \leq l_d-1$, we have $deg_{G''}(v_i) \geq (n-4)-((n-3)-d) = d-1$.

Thus, $G[V_d \cup \{v_{l_d}\}]$ is a graph removing some (none, one, or more) *vertex-disjoint-edges* of the complete graph $K_{V_d\cup\{v_{l_d}\}}$.

Case 2. If $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$, then $deg_{G''}(v_{n-2}) = (n-4)-d$. We have proved in Case 2.2 of (2) that $G[VLD \cup \{v_{l_d}\}]$ is a graph removing some (none, one, or more) *vertex-*

disjoint-edges of the complete graph $K_{VLD \cup \{v_{l_d}\}}$.
Since $deg_{G''}(v_{n-2}) + deg_{G''}(v_i) \geq n-4$ holds for any $1 \leq i \leq l_d - 1$, we have $deg_{G''}(v_i) \geq (n-4) - ((n-4)-d) = d$.

Therefore, $G[V_d \cup \{v_{l_d}\}]$ is a complete graph $K_{V_d \cup \{v_{l_d}\}}$. On the basis of these two cases, we can conclude that $G'' \in K'_{V_d} : v_{l_d} : K'_{VLD}$.

Theorem 20: For a graph $G = (V, E)$ with $|G| = |V| = n \geq 7$, if $\kappa(G) \geq 4$ and $\sigma_2(G) \geq n$, then either G is 2-*vertex-fault Hamiltonian* or G belongs to one of the families $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8\}$.

***Proof*:** Note that S_H, T_H, W_H are defined in Definition 15. For any two vertices $x, y \in V$, let $G' = (V', E')$, with $V' = V - \{x\}$, $E' = E - \{(x, s) \mid s \in V\}$, and $|V'| = n' = n-1$.
Case 1. If G is 1-*vertex-fault Hamiltonian*, G' must have a Hamiltonian cycle with $\kappa(G') \geq 3$, and $deg_{G'}(u) + deg_{G'}(v) \geq n-2 = n'-1$ for any nonadjacent pair $\{u, v\} \subset V(G')$, based on previous discussion.

Delete a vertex y from V' to obtain a graph G'', where $G'' = (V'', E'')$, with $V'' = V' - \{y\}$, $E'' = E' - \{(t, y) | t \in V'\}$, $|V''| = n-2 = n'-1 = n''$, $\kappa(G'') \geq 2$, and $\sigma_2(G'') \geq n-4 = n'-3 = n''-2$. Then we have two possibilities: G'' is Hamiltonian or it is not.
Case 1.1. G'' is Hamiltonian. Then G is 2-*vertex*-fault Hamiltonian.
Case 1.2. G'' is not Hamiltonian.

In this case, G'' must contain a Hamiltonian path $HP = \langle v_1, v_2, v_3, \ldots, v_{n-2} \rangle$. Clearly, $(v_1, v_{n-2}) \notin E(G'')$; otherwise, G'' is Hamiltonian. The condition "$\kappa(G) \geq 4$" indicates that $deg_G(v_1) \geq 4$. Let $N_{G''}(v_1) = \{v_{l_1}, v_{l_2}, v_{l_3}, ..., v_{l_d}\}$ where $2 = l_1 < l_2 < l_3 < \ldots < l_d$.

According to Lemma 12, $(v_{n-2}, v_{l_r-1}) \notin E$ for all l_r with $1 \leq r \leq d$; otherwise G'' is Hamiltonian. By Lemma 16, $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$ or $n-3$. Note that v_{l_d} is the neighbor of v_1 that has the largest subscript. It follows that there are two possibilities: $l_d = n-3$ and $l_d < n-3$.
Case 1.2.1 $l_d = n-3$; that is, $(v_1, v_{n-3}) \in E(G'')$.

Since $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$ or $n-3$, we have two possibilities to consider: $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-3$ and $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-4$.
Case 1.2.1.1 $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n-3$.

On the basis of $(n-5)/2 \leq deg_{G''}(v_{n-2}) \leq (n-3)/2$ from Lemma 17, we have three possibilities: $deg_{G''}(v_{n-2}) = (n-3)/2$ for n is odd, $deg_{G''}(v_{n-2}) = (n-4)/2$ for n is even, and $deg_{G''}(v_{n-2}) = (n-5)/2$ for n is odd.
Case 1.2.1.1.1 $deg_{G''}(v_{n-2}) = (n-3)/2$ for n is odd.

With $deg_{G''}(v_{n-2}) = (n-3)/2$ and Lemma 14 – there are no two consecutive vertices in the Hamiltonian path HP in $N_{G''}(v_{n-2})$, we have $N_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-5}, \ldots, v_4, v_2\} = T_H$. And, by Lemma 16, we have $S_H = \{v_{n-4}, v_{n-6}, \ldots, v_3, v_1\} = \bar{N}_{G''}(v_{n-2})$. So, $N_{G''}(v_1) = \{v_{n-3}, v_{n-5}, \ldots, v_4, v_2\} = \{v_{l_r} | l_r = 2 \times r, 1 \leq r \leq (n-3)/2\} = N_{G''}(v_{n-2})$. See Fig. 13.

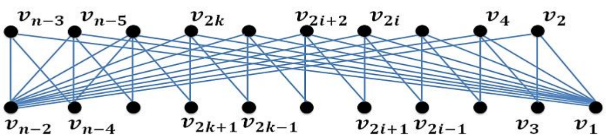


Fig. 13. An illustration of Case 1.2.1.1.1 for $N_{G''}(v_1) = N_{G''}(v_{n-2})$.

By Lemma 17, $\forall v_{m_t-1} \in \bar{N}_{G''}(v_{n-2})$, we have $(n-5)/2 \leq deg_{G''}(v_{m_t-1}) \leq (n-3)/2$. In this case,

$\forall v_{2\times i-1} \in \overline{N}_{G''}(v_{n-2})$ for $1 \le i \le (n-3)/2$, we have $(n-5)/2 \le deg_{G''}(v_{2\times i-1}) \le (n-3)/2$ which obviously has two subcases to consider.

Subcase 1.2.1.1.1.1 $\forall v_{2\times i-1} \in \overline{N}_{G''}(v_{n-2})$, $deg_{G''}(v_{2\times i-1}) = (n-3)/2$.

With $deg_{G''}(v_{2\times i-1}) = (n-3)/2$ and Lemma 14 – the vertices in $\overline{N}_{G''}(v_{n-2}) \cup \{v_{n-2}\}$ are mutually nonadjacent, we have $N_{G''}(v_{2\times i-1}) = \{v_{l_r} | l_r = 2 \times r,\ 1 \le r \le (n-3)/2\}$ for all $1 \le i \le (n-3)/2$. Let $V(H_{(n-3)/2}) = N_{G''}(v_{n-2})$, and $\overline{N}_{G''}(v_{n-2}) \cup \{v_{n-2}\} = V(\overline{K_{(n-1)/2}})$. Then G'' can be written as $H_{(n-3)/2} \vee \overline{K_{(n-1)/2}}$. For $n = 9$, the graph G'' is illustrated in Fig. 8 (c). Since the number of components of $G'' - \{v_{n-3}, v_{n-5}, \ldots, v_4, v_2\}$ is greater than $|\{v_{n-3}, v_{n-5}, \ldots, v_4, v_2\}|$, G'' is not Hamiltonian by Theorem 3. There are two possibilities to reconstruct G from G'', as shown below.

Adding two vertices x, y to the graph G'', we can obtain $G = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}})$ with $\sigma_2(H_{(n+1)/2}) \ge 1$ when $n \ge 9$ and $\kappa(H_4) \ge 1$ when $n = 7$. Hence $G \in \eta_4$.

Adding two vertices x, y to the graph G'', and deleting one edge (v_α, v_θ) where $v_\alpha \in \{x, y\}$ and $v_\theta \in V(\overline{K_{(n-1)/2}})$, with $deg_{H_{(n+1)/2}}(v_\alpha) \ge 2$, $\sigma_2(H_{(n+1)/2}) \ge 1$, and $n \ge 9$, we have $G = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}}) - (v_\alpha, v_\theta)$. Thus, $G \in \eta_3$.

Subcase 1.2.1.1.1.2 There exists an element $v_{2\times t-1} \in \overline{N}_{G''}(v_{n-2})$ with $deg_{G''}(v_{2\times t-1}) = (n-5)/2$.

Based on Lemma 17, the degrees of all other $v_{2\times s-1} \in \overline{N}_{G''}(v_{n-2})$ must be equal to $(n-3)/2$. So, G'' is of the form in Fig. 6 (b) when $n = 9$, which is not Hamiltonian. Let $v_{2\times t-1} = v_\theta$, G'' can be written as $H_{(n-3)/2} \vee \overline{K_{(n-1)/2}} - (v_\alpha, v_\theta)$, where $v_\alpha \in V(H_{(n-3)/2})$. By adding two vertices x, y to G'', we can find $G = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}}) - (v_\alpha, v_\theta)$, with $deg_{H_{(n+1)/2}}(v_\alpha) \ge 2$, and $\sigma_2(H_{(n+1)/2}) \ge 1$. Consequently, $G \in \eta_3$. See Fig. 6 (a).

Case 1.2.1.1.2 $deg_{G''}(v_{n-2}) = (n-4)/2$, n is even.

On the basis of $deg_{G''}(v_{n-2}) = (n-4)/2$ and Lemma 14 – there are no two consecutive vertices of the Hamiltonian path HP in $N_{G''}(v_{n-2})$, we have $N_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-5}, \ldots, v_5, v_3\} = T_H$, and $\overline{N}_{G''}(v_{n-2}) = \{v_{n-4}, v_{n-6}, \ldots, v_4, v_2\}$. By Lemma 16, $S_H = \{v_{n-4}, v_{n-6}, \ldots, v_4, v_2, v_1\}$; therefore, $N_{G''}(v_1) = \{v_{n-3}, v_{n-5}, \ldots, v_5, v_3, v_2\}$ and $deg_{G''}(v_1) = (n-2)/2$. See Fig. 14.

By Lemmas 14 and 17, the vertices in $\overline{N}_{G''}(v_{n-2})$ are all mutually nonadjacent, and $(n-4)/2 \le deg_{G''}(v_{m_t-1}) \le (n-2)/2$. There are two subcases to consider.

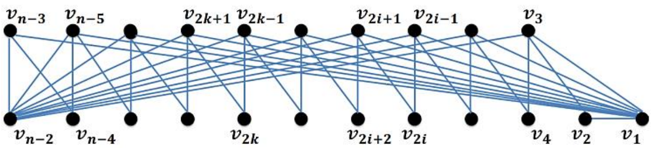


Fig. 14. An illustration of Case 1.2.1.1.2 for $deg_{G''}(v_{n-2}) = (n-4)/2$.

Subcase 1.2.1.1.2.1 There exists one vertex $v_{m_t-1} \in \overline{N}_{G''}(v_{n-2})$ with $deg_{G''}(v_{m_t-1}) = (n-2)/2$.

By Lemma 17, except vertex v_{m_t-1}, all other vertices $v_{m_s-1} \in \overline{N}_{G''}(v_{n-2})$ have $deg_{G''}(v_{m_s-1}) = (n-4)/2$. Since, as discussed previously in Lemma 14, all elements belonging to $\overline{N}_{G''}(v_{n-2}) \cup \{v_{n-2}\}$ are mutually nonadjacent, we have $N_{G''}(v_{m_s-1}) = \{v_{n-3}, v_{n-5}, \ldots, v_5, v_3\} = N_{G''}(v_{n-2})$ and $deg_{G''}(v_{m_s-1}) = (n-4)/2$ for each $v_{m_s-1} \in \overline{N}_{G''}(v_{n-2}) - \{v_2\}$. As for v_2, we have $N_{G''}(v_2) = \{v_{n-3}, v_{n-5}, \ldots, v_5, v_3, v_1\}$ and $deg_{G''}(v_2) = (n-2)/2$. Let $V(H_{(n-4)/2}) = N_{G''}(v_{n-2})$, $V(\overline{K_{(n-4)/2}}) = \overline{N}_{G''}(v_{n-2}) \cup \{v_{n-2}\} - \{v_2\}$. Then G'' can be written as $H_{(n-4)/2} \vee (\overline{K_{(n-4)/2}} \cup K_2)$, where $V(K_2) = \{v_1, v_2\}$. Thus G'' is of the form in Fig. 9 (b) when $n = 10$. The number of components of $G'' - \{v_{n-3}, v_{n-5}, \ldots, v_5, v_3\}$ is greater than $|\{v_{n-3}, v_{n-5}, \ldots, v_5, v_3\}|$. Consequently, by Theorem 3, G'' is not Hamiltonian. There are three possibilities to reconstruct G from G'', as shown below.

Subcase 1.2.1.1.2.1.1

Adding two vertices x, y to G'' such that $G = ((H_{(n-4)/2} : x : y) \vee (\overline{K_{(n-4)/2}} \cup K_2)) = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2)$, we have $\delta(G) = (n/2)$ and $\sigma_2(G) = n$.

In the graph G, if $H_{n/2} = \overline{K_{n/2}}$, then $\overline{K_{n/2}} \vee (\overline{K_{(n-4)/2}} \cup K_2)$ is a special case of $\overline{K_{n/2}} \vee H_{n/2}$. Otherwise, we will have $G = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) = \eta_5$, in which the complete graph in $(\overline{K_{(n-4)/2}} \cup K_2)$ is with $V(K_2) = \{v_1, v_2\}$. This case shows that $n \geq 8$ is required for ensuring $\kappa(G) \geq 4$. The graph η_5 for $n = 10$ is illustrated in Fig. 9 (a).

Subcase 1.2.1.1.2.1.2

We can delete one edge (v_α, v_θ) from $H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2)$, where $v_\alpha \in \{x, y\}$ and $v_\theta \in \{v_1, v_2\}$, with $deg_{H(n/2)}(v_\alpha) \geq 1$. Then we obtain the graph $G = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_\theta)$, which must belong to η_6. For $n = 10$, the graph G is illustrated in Fig. 9 (c).

Subcase 1.2.1.1.2.1.3

We can delete two edges (v_α, v_θ), and $(v_\omega, v_\varepsilon)$ from $H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2)$, where v_θ, $v_\varepsilon \in \{v_1, v_2\}$, $v_\theta \neq v_\varepsilon$; $v_\alpha, v_\omega \in \{x, y\}$, $v_\alpha \neq v_\omega$ with $deg_{H(n/2)}(v_\alpha) \geq 1$ and $deg_{H(n/2)}(v_\omega) \geq 1$. Then we obtain $G = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_\theta) - (v_\omega, v_\varepsilon)$, which must belong to η_7. For $n = 10$, the graph G is illustrated in Fig. 9 (d).

Subcase 1.2.1.1.2.2 For each element $v_{m_t-1} \in N^-_{G''}(v_{n-2})$, $deg_{G''}(v_{m_t-1}) = (n-4)/2$.

In this case, $deg_{G''}(v_2) = (n-4)/2$. There must be an element $v_\alpha \in N_{G''}(v_{n-2})$, $(v_\alpha, v_2) \notin E(G'')$. Let $V(H_{(n-4)/2}) = N_{G''}(v_{n-2})$ and $V(\overline{K_{(n-4)/2}}) = N^-_{G''}(v_{n-2}) \cup \{v_{n-2}\} - \{v_2\}$. Then G'' can be written as $H_{(n-4)/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_2)$, where $V(K_2) = \{v_1, v_2\}$. Thus G'' is of the form in Fig. 10 (c) for $n = 12$. There are two possibilities to reconstruct G from G'', as shown below.

Subcase 1.2.1.1.2.2.1

Adding two vertices x, y to G'', we can obtain $G = ((H_{(n-4)/2} : x : y) \vee (\overline{K_{(n-4)/2}} \cup K_2)) - (v_\alpha, v_2) = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_2)$, in which the complete graph in $(\overline{K_{(n-4)/2}} \cup K_2)$ is with $V(K_2) = \{v_1, v_2\}$. To ensure $\sigma_2(G) = n$, we must have $deg_{H(n/2)}(v_\alpha) \geq 1$, which implies that $G \in \eta_6$. This case shows that $n \geq 8$ is required for ensuring $\kappa(G) \geq 4$. The graph G of $n = 12$ is shown in Fig. 10 (b).

Subcase 1.2.1.1.2.2.2

We can delete one edge (v_ω, v_1) from $H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_2)$, where $v_\omega \in \{x, y\}$ with $deg_{H(n/2)}(v_\omega) \geq 1$. Then we have $G = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_2) - (v_\omega, v_1)$, which must belong to η_7. This case shows that $n \geq 8$ is required for ensuring $\kappa(G) \geq 4$. The graph G of $n = 12$ is shown in Fig. 10 (d).

Case 1.2.1.1.3 $deg_{G''}(v_{n-2}) = (n-5)/2$, n is odd.

In this case, $deg_{G''}(v_1) = (n-1)/2$. By $(v_1, v_{m_t+1}) \notin E(G)$ from Lemma 12 and $\forall v_{m_t+1} \in N^+_{G''}(v_{n-2}) - \{v_{n-2}\}$ from Lemma 17, we have $deg_{G''}(v_{m_t+1}) = (n-3)/2$, which leads to $deg_{G''}(v_1) + deg_{G''}(v_{m_t+1}) = n-2$. Connecting v_1 to v_{m_t+1}, by Lemma 14, we can see that $G'' + (v_1, v_{m_t+1})$ is Hamiltonian. But, by Theorem 2, G'' is Hamiltonian too. This is a contradiction.

Case 1.2.1.2 $deg_{G''}(v_1) + deg_{G''}(v_{n-2}) = n - 4$.

Case 1.2.1.2.1 $deg_{G''}(v_{n-2}) = (n-3)/2$, n is odd.

In this case, $deg_{G''}(v_1) = (n-5)/2$. This is a special case of $v_\theta = v_1$ in Subcase 1.2.1.1.1.2. Similarly, we have $G'' = H_{(n-3)/2} \vee \overline{K_{(n-1)/2}} - (v_\alpha, v_1)$, where $v_\alpha \in V(H_{(n-3)/2})$. By adding two vertices to G'', we obtain $G = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}}) - (v_\alpha, v_1)$, in which $\sigma_2(H_{(n+1)/2}) \geq 1$ and $deg_{H(n+1)/2}(v_\alpha) \geq 2$. It can be seen that $G \in \eta_3$.

Case 1.2.1.2.2 $deg_{G''}(v_{n-2}) = (n-4)/2$, n is even.

This case is similar to Case 1.2.1.1.2. In this case, $deg_{G''}(v_1) = (n-4)/2$. Because of $deg_{G''}(v_{n-2}) = (n-4)/2$ and Lemma 14 – there are no two consecutive vertices of the Hamiltonian path HP in $N_{G''}(v_{n-2})$, we have $N_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-5}, \ldots, v_5, v_3\} = T_H$, which shows $N^-_{G''}(v_{n-2}) = \{v_{n-4}, v_{n-6}, \ldots, v_4, v_2\}$. By $deg_{G''}(v_1) = (n-4)/2$ and Lemma 16, we have $S_H = \{v_{n-4}, v_{n-6}, \ldots, v_4, v_2, v_1\} - \{v_{2k}\}$ and $W_H = \{v_{2k}\}$ with $v_{2k} \neq v_{n-4}$, which result in $N_{G''}(v_1) = \{v_{n-3}, v_{n-5}, \ldots, v_5, v_3, v_2\} - \{v_{2k+1}\}$. Let $v_{2k+1} = v_\alpha$. Then, there are two subcases in this case.

Subcase 1.2.1.2.2.1 There exists one vertex $v_{m_t-1} \in N^-_{G''}(v_{n-2})$ with $deg_{G''}(v_{m_t-1}) = (n-2)/2$.

The vertex v_{m_t-1} must be v_2 because $N_{G''}(v_2) = \{v_{n-3}, v_{n-5}, \ldots, v_5, v_3, v_1\}$ as described in Subcase 1.2.1.1.2.1. The graph G'' can be written as $H_{(n-4)/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_1)$. The graph G'' of $n = 12$ is shown in Fig. 10 (f). There are two possibilities to reconstruct G from G'', as shown below.

Subcase 1.2.1.2.2.1.1

We can add two vertices x, y to G'' such that $G = ((H_{(n-4)/2} : x : y) \vee (\overline{K_{(n-4)/2}} \cup K_2)) - (v_\alpha, v_1) = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_1)$, where the complete graph in $(\overline{K_{(n-4)/2}} \cup K_2)$ is with (v_1, v_2). To ensure $\sigma_2(G) = n$, we must have $deg_{H_{(n/2)}}(v_\alpha) \geq 1$, which implies that $G \in \eta_6$, as shown in Fig. 10 (a) for $n = 12$.

Subcase 1.2.1.2.2.1.2

We can delete one edge (v_ϖ, v_2) from $H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_1)$, where $v_\varpi \in \{x, y\}$ with $deg_{H(n/2)}(v_\varpi) \geq 1$. Then we have $G = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_1) - (v_\varpi, v_2)$, which implies that $G \in \eta_7$, as shown in Fig. 10 (e) for $n = 12$.

Subcase 1.2.1.2.2.2 For each element $v_{m_t-1} \in N^-_{G''}(v_{n-2})$, $deg_{G''}(v_{m_t-1}) = (n-4)/2$.

In this case, $deg_{G''}(v_2) = (n-4)/2$. There must be an element $v_\varpi \in N_{G''}(v_{n-2})$, $v_\varpi \neq v_\alpha$, $v_\varpi \neq v_3$ and $(v_\varpi, v_2) \notin E(G'')$. Let $V(H_{(n-4)/2}) = N_{G''}(v_{n-2})$, $V(\overline{K_{(n-4)/2}}) = N^-_{G''}(v_{n-2}) \cup \{v_{n-2}\} - \{v_2\}$. Then G'' can be written as $H_{(n-4)/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_1) - (v_\varpi, v_2)$. The graph G'' of $n = 12$ is shown in Fig. 11 (b). Adding two vertices x, y to G'', we can obtain $G = (H_{(n-4)/2} : x : y) \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_1) - (v_\varpi, v_2) = H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_1) - (v_\varpi, v_2)$, in which the complete graph in $(\overline{K_{(n-4)/2}} \cup K_2)$ is with $V(K_2) = \{v_1, v_2\}$. To ensure $\sigma_2(G) = n$, we must have $deg_{H_{(n/2)}}(v_\alpha) \geq 1$ and $deg_{H_{(n/2)}}(v_\varpi) \geq 1$, which implies that $H_{n/2} \vee (\overline{K_{(n-4)/2}} \cup K_2) - (v_\alpha, v_1) - (v_\varpi, v_2) \in \eta_7$, as shown in Fig. 11 (a) for $n = 12$.

Case 1.2.1.2.3 $deg_{G''}(v_{n-2}) = (n-5)/2$, n is odd.

By Lemma 16, $deg_{G''}(v_1) = (n-3)/2$. There are two cases to consider: "$n > 9$" and "$n = 9$".

Case 1.2.1.2.3.1 $n > 9$.

Subcase 1.2.1.2.3.1.1 $N_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-5}, \ldots, v_{2(s+2)}, v_{2(s+1)}, v_{2s}, v_{2(s-1)}, v_{2(s-2)}, \ldots, v_4, v_2\} - \{v_{2s}\}$ where $v_{2s} \neq v_{n-3}$, that is $N_{G''}(v_{n-2}) = \{v_{l_r} | l_r = 2 \times r, 1 \leq r \leq (n-3)/2\} - \{v_{2s}\} = T_H$.

Subcase 1.2.1.2.3.1.1.1 $v_{2s} \neq v_2$, that is, $s \neq 1$.

It can be seen that

$V(G'') - \{v_{n-2}\} - T_H = S_H \cup W_H = \{v_{n-4}, v_{n-6}, \ldots, v_{2s+5}, v_{2s+3}, v_{2s+1}, v_{2s}, v_{2s-1}, v_{2s-3}, v_{2s-5}, \ldots, v_3, v_1\}$;

$N^-_{G''}(v_{n-2}) = \{v_{n-4}, v_{n-6}, \ldots, v_{2s+3}, v_{2s+1}, v_{2s-3}, v_{2s-5}, \ldots, v_3, v_1\}$;

$N^+_{G''}(v_{n-2}) = \{v_{n-2}, v_{n-4}, v_{n-6}, \ldots, v_{2s+5}, v_{2s+3}, v_{2s-1}, v_{2s-3}, \ldots, v_5, v_3\}$;

$N^-_{G''}(v_{n-2}) \cap N^+_{G''}(v_{n-2}) = \{v_{n-4}, v_{n-6}, \ldots, v_{2s+5}, v_{2s+3}, v_{2s-3}, v_{2s-5}, \ldots, v_5, v_3\}$;

$N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2}) = \{v_{n-2}, v_{n-4}, v_{n-6}, \ldots, v_{2s+5}, v_{2s+3}, v_{2s+1}, v_{2s-1}, v_{2s-3}, v_{2s-5}, \ldots, v_5, v_3, v_1\}$;

$V(G'') - N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-5}, \ldots, v_{2(s+2)}, v_{2(s+1)}, v_{2s}, v_{2(s-1)}, v_{2(s-2)}, \ldots, v_4, v_2\}$
$= \{v_{l_r} | l_r = 2 \times r, 1 \leq r \leq (n-3)/2\}$, and $|V(G'') - N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2})| = (n-3)/2$.

For each vertex $v_\theta \in N^-_{G''}(v_{n-2}) \cap N^+_{G''}(v_{n-2})$, we have $deg_{G''}(v_\theta) = (n-3)/2$ by Lemma 17; by Lemma 14, $N^-_{G''}(v_{n-2}) \cup \{v_{n-2}\}$ are mutually nonadjacent to each other, and $N^+_{G''}(v_{n-2}) \cup \{v_1\}$ are mutually nonadjacent to each other too; therefore, we have $N_{G''}(v_\theta) \subseteq V(G'') -$

$N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2}) = N_{G''}(v_{n-2}) \cup \{v_{2s}\}$. Since $|V(G'')-N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2})| = (n-3)/2$, we conclude that $N_{G''}(v_\theta) = N_{G''}(v_{n-2}) \cup \{v_{2s}\}$.

As for the vertex $v_{2s-1} \in N^+_{G''}(v_{n-2})$, we have $deg_{G''}(v_{2s-1}) = (n-3)/2$ by Lemma 17, and $N_{G''}(v_{2s-1}) \subset V(G'')-N^+_{G''}(v_{n-2})-\{v_1\} = \{v_{n-3}, v_{n-5}, ..., v_{2(s+1)}, v_{2s+1}, v_{2s}, v_{2(s-1)}, ..., v_4, v_2\}$. Note that $v_{2s-3} \in N^-_{G''}(v_{n-2}) \cap N^+_{G''}(v_{n-2})$; therefore, $v_{2s} \in N_{G''}(v_{2s-3})$. If $(v_{2s+1}, v_{2s-1}) \in E(G'')$, then $\langle v_1, v_{n-3}, \searrow, v_{2s+1}, v_{2s-1}, v_{2s}, v_{2s-3}, v_{2s-2}, v_{n-2}, v_{2s-4}, \searrow, v_1\rangle$ is a Hamiltonian cycle, which is a contradiction. Hence $N_{G''}(v_{2s-1}) = V(G'')-N^+_{G''}(v_{n-2})-\{v_1\}-\{v_{2s+1}\} = N_{G''}(v_{\boldsymbol\theta})$. For the vertex $v_{2s+1} \in N^-_{G''}(v_{n-2})$, we can prove that $N_{G''}(v_{2s+1}) = N_{G''}(v_{\boldsymbol\theta})$ in a similar manner.

Note that $N^+_{G''}(v_{n-2}) \cup \{v_1\} \cup \{v_{2s+1}\} = N^-_{G''}(v_{n-2}) \cup \{v_{n-2}\} \cup \{v_{2s-1}\}$; the vertices belonging to $N^-_{G''}(v_{n-2}) \cup \{v_{n-2}\}$ are mutually nonadjacent to each other; the vertices belonging to $N^+_{G''}(v_{n-2}) \cup \{v_1\}$ are mutually nonadjacent to each other; $v_{2s-1} \in N^+_{G''}(v_{n-2})$ and $v_{2s+1} \in N^-_{G''}(v_{n-2})$; and $(v_{2s+1}, v_{2s-1}) \notin E(G'')$; hence, we can conclude that the vertices in $N^-_{G''}(v_{n-2}) \cup \{v_{n-2}\} \cup \{v_{2s-1}\}$ are mutually nonadjacent to each other too.

Subcase 1.2.1.2.3.1.1.1.1 $v_{2s} \in S_H$.

If $v_{2s} \in S_H$, this implies that $v_{2s+1} \in N_{G''}(v_1)$. For a vertex $v_{\boldsymbol\theta} \in N^-_{G''}(v_{n-2}) \cap N^+_{G''}(v_{n-2})$, w.l.o.g. let $\theta > 2s$; we have $\langle v_1, \nearrow, v_{2s}, v_{\boldsymbol\theta}, \nearrow, v_{n-2}, v_{\boldsymbol\theta-1}, \searrow, v_{2s+1}, v_1\rangle$ is a Hamiltonian cycle. This is a contradiction.

Subcase 1.2.1.2.3.1.1.1.2 $v_{2s} \notin S_H$, that is, $S_H = \{v_{n-4}, v_{n-6}, ..., v_{2s+1}, v_{2s-1}, ..., v_3, v_1\}$ and $W_H = \{v_{2s}\}$.

We have $N_{G''}(v_1) = \{v_{n-3}, v_{n-5}, ..., v_4, v_2\} = \{v_{l_r} | l_r = 2 \times r,\ 1 \le r \le (n-3)/2\} = N_{G''}(v_{n-2}) \cup \{v_{2s}\}$, and $N_{G''}(v_{n-2}) \subset N_{\boldsymbol G''}(v_1)$. This is a special case of $v_\theta = v_{n-2}$ in Subcase 1.2.1.1.1.2.

Let $V(H_{(n-3)/2}) = N_{\boldsymbol G''}(v_{n-2}) \cup \{v_{2s}\}$, $V(\overline{K_{(n-1)/2}}) = N^-_{G''}(v_{n-2}) \cup \{v_{2\times s-1}\} \cup \{v_{n-2}\}$, and $v_{2s} = v_\alpha$; then G'' can be written as $H_{(n-3)/2} \vee \overline{K_{(n-1)/2}} - (v_\alpha, v_{n-2})$. Adding two vertices to G'', we can find the graph $G = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}}) - (v_\alpha, v_{n-2})$, where $\sigma_2(H_{(n+1)/2}) \ge 1$ and $deg_{H_{(n+1)/2}}(v_\alpha) \ge 2$. Hence, $G \in \eta_3$.

Subcase 1.2.1.2.3.1.1.2 $v_{2s} = v_2$, that is, $s = 1$.

We have $N_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-5}, \ldots, v_8, v_6, v_4\}$;

$V(G'')-\{v_{n-2}\}-T_H = S_H \cup W_H = \{v_{n-4}, v_{n-6}, \ldots, v_7, v_5, v_3, \boldsymbol{v_2}, v_1\}$;

$N^-_{G''}(v_{n-2}) = \{v_{n-4}, v_{n-6}, \ldots, v_9, v_7, v_5, v_3\}$; $N^+_{G''}(v_{n-2}) = \{v_{n-2}, v_{n-4}, v_{n-6}, \ldots, v_{11}, v_9, v_7, v_5\}$;

$N^-_{G''}(v_{n-2}) \cap N^+_{G''}(v_{n-2}) = \{v_{n-4}, v_{n-6}, v_{n-8}, \ldots, v_{11}, v_9, v_7, v_5\}$;

$N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2}) = \{v_{n-2}, v_{n-4}, v_{n-6}, \ldots, v_{11}, v_9, v_7, v_5, v_3\}$;

$V(G'')-N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-5}, \ldots, v_8, v_6, v_4, v_2, v_1\}$.

In a similar manner, we can find that for each $v_\theta \in N^-_{G''}(v_{n-2}) \cap N^+_{G''}(v_{n-2})$, $N_{G''}(v_{\boldsymbol\theta}) = V(G'')-N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2})-\{v_1\} = N_{G''}(v_{n-2}) \cup \{v_2\}$; $(v_1, v_3) \notin E(G'')$; $N_{G''}(v_1) = N_{G''}(v_3) = N_{G''}(v_\theta)$, as shown in Subcase 1.2.1.2.3.1.1.1. We can obtain that $G'' = H_{(n-3)/2} \vee \overline{K_{(n-1)/2}} - (v_2, v_{n-2})$ and $G = (H_{(n+1)/2} \vee \overline{K_{(n-1)/2}}) - (v_2, v_{n-2})$, where $\sigma_2(H_{(n+1)/2}) \ge 1$ and $deg_{H_{(n+1)/2}}(v_2) \ge 2$ in a similar way, as shown in Subcase 1.2.1.2.3.1.1.1.2.

Subcase 1.2.1.2.3.1.2 $N_{G''}(v_{n-2}) \not\subseteq \{v_{n-3}, v_{n-5}, ..., v_4, v_2\}-\{v_{2s}\}$

Since $v_{n-3} \in N_{G''}(v_{n-2})$, v_{n-4} must not belong to $N_{G''}(v_{n-2})$. It follows that we will examine each vertex sequentially. The next vertex to be examined is v_{n-5}.

Subcase 1.2.1.2.3.1.2.1 $v_{n-5} \notin N_{G''}(v_{n-2})$.

With $deg_{G''}(v_{n-2}) = (n-5)/2$ and Lemma 14 – there are no two consecutive vertices in the Hamiltonian path HP in $N_{G''}(v_{n-2})$, we have either $N_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-6}, v_{n-8}, \ldots, v_5, v_3\}$ or $N_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-7}, v_{n-9}, \ldots, v_4, v_2\}$. Based on "$N_{G''}(v_{n-2}) \not\subseteq \{v_{n-3}, v_{n-5}, ..., v_4, v_2\}-\{v_{2s}\}$", we must have $N_{G''}(v_{n-2}) = \{v_{n-3}, v_{n-6}, v_{n-8}, \ldots, v_5, v_3\}$, which leads to $N^-_{G''}(v_{n-2}) = \{v_{n-4}, v_{n-7}, v_{n-9}, \ldots, v_4, v_2\}$ and $N^+_{G''}(v_{n-2}) = \{v_{n-2}, v_{n-5}, v_{n-7}, v_{n-9}, ..., v_6, v_4\}$. It can be seen that $v_{n-7} \in N^-_{G''}(v_{n-2})$,

$v_{n-7} \in N^-_{G''}(v_{n-2})$, and $N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2}) \cup \{v_1\} = \{v_{n-2}, v_{n-4}, v_{n-5}, v_{n-7}, v_{n-9}, \ldots, v_6, v_4, v_2, v_1\}$. Based on Lemma 14 – the vertices in $N^-_{G''}(v_{n-2}) \cup \{v_{n-2}\}$ are mutually nonadjacent and the vertices in $N^+_{G''}(v_{n-2}) \cup \{v_1\}$ are mutually nonadjacent, the neighbors of v_{n-7} are in the set of $\{v_1, v_2, v_3, v_4, \ldots, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}\} - N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2}) \cup \{v_1\} = \{v_{n-3}, v_{n-6}, v_{n-8}, \ldots, v_5, v_3\}$, from which we can see that $|\{v_{n-3}, v_{n-6}, v_{n-8}, \ldots, v_5, v_3\}| = (n-5)/2$. However, by Lemma 17, $deg_{G''}(v_{n-7}) = (n-3)/2$, which is a contradiction. The graph G'' for $n = 23$ is shown in Fig. 15, in which $N_{G''}(v_{21}) = \{v_{20}, v_{17}, v_{15}, \ldots, v_5, v_3\}$, $N^-_{G''}(v_{21}) = \{v_{19}, v_{16}, v_{14}, \ldots, v_4, v_2\}$ and $N^+_{G''}(v_{21}) = \{v_{21}, v_{18}, v_{16}, v_{14}, \ldots, v_6, v_4\}$, $v_{16} \in N^-_{G''}(v_{21})$, $v_{16} \in N^+_{G''}(v_{21})$, $|\{v_1, \ldots, v_{n-2}\} - N^-_{G''}(v_{n-2}) \cup N^+_{G''}(v_{n-2}) \cup \{v_1\}| = |\{v_{20}, v_{17}, v_{15}, \ldots, v_5, v_3\}| = 9$.

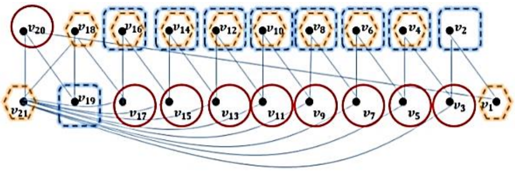


Fig. 15. G'' for $n = 23$.

Subcase 1.2.1.2.3.1.2.2 $v_{n-5} \in N_{G''}(v_{n-2})$.

For $v_{n-5} \in N_{G''}(v_{n-2})$, v_{n-6} must not belong to $N_{G''}(v_{n-2})$. Therefore, the next vertex to be examined is v_{n-7}. If $v_{n-7} \notin N_{G''}(v_{n-2})$, this, similar to Subcase 1.2.1.2.3.1.2.1, will result in a contradiction. If $v_{n-7} \in N_{G''}(v_{n-2})$, v_{n-8} must not belong to $N_{G''}(v_{n-2})$. Hence, the next vertex to be examined is v_{n-9}. Continuing this examining process, we will arrive at either a case of Subcase 1.2.1.2.3.1.1 or a contradiction.

Case 1.2.1.2.3.2 $n = 9$.

There is a Hamiltonian path $\langle v_1, v_2, v_3, v_4, v_5, v_6, v_7\rangle$, where $(v_1, v_6) \in E(G'')$, $deg_{G''}(v_1) = (9-3)/2 = 3$, and $deg_{G''}(v_7) = (9-5)/2 = 2$.

Subcase 1.2.1.2.3.2.1 $N_{G''}(v_7) = \{v_2, v_4, v_6\} - \{v_{2s}\}$ where $v_{2s} \neq v_6$.

Subcase 1.2.1.2.3.2.1.1 $N_{G''}(v_7) = \{v_2, v_6\}$.

In this case, we have $V(G'') - \{v_7\} - T_H = S_H \cup W_H = \{v_1, v_3, v_4, v_5\}$, $N^-_{G''}(v_7) = \{v_1, v_5\}$, $N^+_{G''}(v_7) = \{v_3, v_7\}$, by Lemma 17, we have $deg_{G''}(v_3) = deg_{G''}(v_5) = 3$. It can be seen that $N_{G''}(v_1) \subset V(G'') - N^-_{G''}(v_7) - \{v_7\} = \{v_2, v_3, v_4, v_6\}$. If $v_3 \in N_{G''}(v_1)$, then v_2 must belong to S_H. However, $v_2 \notin S_H \cup W_H$; hence, $v_2 \notin S_H$. Consequently, $v_3 \notin N_{G''}(v_1)$ and $N_{G''}(v_1) = \{v_2, v_4, v_6\}$. For $N_{G''}(v_3) \subset V(G'') - N^+_{G''}(v_7) - \{v_1\} = \{v_2, v_4, v_5, v_6\}$, if $(v_3, v_5) \in E(G'')$, then $\langle v_1, v_4, v_3, v_5, v_6, v_7, v_2, v_1\rangle$ is a Hamiltonian cycle. This is a contradiction. Therefore, $N_{G''}(v_3) = \{v_2, v_4, v_6\}$. Moreover, for $N_{G''}(v_5) \subset V(G'') - N^-_{G''}(v_7) - \{v_7\} = \{v_2, v_3, v_4, v_6\}$, and $(v_3, v_5) \notin E(G'')$, then $N_{G''}(v_5) = \{v_2, v_4, v_6\}$. Obviously, the four vertices v_1, v_3, v_5, v_7 are mutually nonadjacent to each other. Let $V(H_3) = N_{G''}(v_7) \cup \{v_4\}$, $V(\overline{K}_4) = N^-_{G''}(v_7) \cup \{v_3\} \cup \{v_7\}$; then G'' can be written as $H_3 \vee \overline{K}_4 - (v_4, v_7)$. See Fig. 3 (a). Since the number of components of $G'' - H_3$ is greater than $|H_3|$, G'', by Theorem 3, is not Hamiltonian. By adding two vertices to G'', the graph G can be found as: $(H_5 \vee \overline{K}_4) - (v_4, v_7)$, $\sigma_2(H_5) \geq 1$, and $deg_{H_5}(v_4) \geq 2$. See Fig. 3 (b). Hence, $G \in \eta_3$.

Subcase 1.2.1.2.3.2.1.2 $N_{G''}(v_7) = \{v_4, v_6\}$.

In this case, we have $V(G'') - \{v_7\} - T_H = S_H \cup W_H = \{v_1, v_2, v_3, v_5\}$, $N^-_{G''}(v_7) = \{v_3, v_5\}$, $N^+_{G''}(v_7) = \{v_5, v_7\}$, and, by Lemma 17, $deg_{G''}(v_3) = deg_{G''}(v_5) = 3$. In addition, $N^-_{G''}(v_7) \cap N^+_{G''}(v_7) = \{v_5\}$, $N^-_{G''}(v_7) \cup N^+_{G''}(v_7) = \{v_3, v_5, v_7\}$, and $N_{G''}(v_5) = V(G'') - N^-_{G''}(v_7) \cup N^+_{G''}(v_7) - \{v_1\} = \{v_2, v_4, v_6\}$. If $v_2 \in S_H$, then $(v_1, v_3) \in E(G'')$, and $\langle v_1, v_3, v_2, v_5, v_4, v_7, v_6, v_1\rangle$ is a Hamiltonian cycle. This is a contradiction. Therefore, $S_H = \{v_1, v_3, v_5\}$, $W_H = \{v_2\}$, and $N_{G''}(v_1) = \{v_2, v_4, v_6\}$. Since $v_3 \in N^-_{G''}(v_7)$, we have $N_{G''}(v_3) \subset V(G'') - N^-_{G''}(v_7) - \{v_7\} = \{v_1, v_2, v_4, v_6\}$. Since $v_3 \notin N_{G''}(v_1)$, we have $N_{G''}(v_3) = \{v_2, v_4, v_6\}$. Obviously, the four vertices v_1, v_3, v_5, v_7 are mutually nonadjacent to each other. Let $V(H_3) = N_{G''}(v_7) \cup \{v_2\}$, $V(\bar{K}_4) = N^-_{G''}(v_7) \cup \{v_1\} \cup \{v_7\}$; then G'' can be written as $H_3 \vee \bar{K}_4 - (v_2, v_7)$. By adding two vertices to G'', the graph G can be found as: $(H_5 \vee \bar{K}_4) - (v_2, v_7)$, $\sigma_2(H_5) \geq 1$, and $deg_{H_5}(v_2) \geq 2$. Hence, $G \in \eta_3$.

Subcase 1.2.1.2.3.2.2 $N_{G''}(v_7) \not\subseteq \{v_2, v_4, v_6\} - \{v_{2s}\}$ where $v_{2s} \neq v_6$.

In this case, we have $N_{G''}(v_7) = \{v_3, v_6\}$ which implies $N^-_{G''}(v_7) = \{v_2, v_5\}$, $N^+_{G''}(v_7) = \{v_4, v_7\}$, and $V(G'') - \{v_7\} - T_H = S_H \cup W_H = \{v_1, v_2, v_4, v_5\}$. Moreover, we can obtain $N_{G''}(v_1) \subset \{v_2, v_3, v_5, v_6\}$; $N_{G''}(v_4) \subset V(G'') - \{v_1\} - N^+_{G''}(v_7) = \{v_2, v_3, v_5, v_6\}$, $N_{G''}(v_2) \subset V(G'') - \{v_7\} - N^-_{G''}(v_7) = \{v_1, v_3, v_4, v_6\}$, and $N_{G''}(v_5) \subset \{v_1, v_3, v_4, v_6\}$. By Lemma 17, $deg_{G''}(v_2) = deg_{G''}(v_5) = deg_{G''}(v_4) = (n-3)/2 = 3$. Hence, we have either $N_{G''}(v_1) = \{v_2, v_3, v_6\}$ or $N_{G''}(v_1) = \{v_2, v_5, v_6\}$; either $N_{G''}(v_4) = \{v_2, v_3, v_5\}$ or $N_{G''}(v_4) = \{v_3, v_5, v_6\}$; either $N_{G''}(v_2) = \{v_1, v_3, v_4\}$ or $N_{G''}(v_2) = \{v_1, v_3, v_6\}$; either $N_{G''}(v_5) = \{v_1, v_4, v_6\}$ or $N_{G''}(v_5) = \{v_3, v_4, v_6\}$. We can find that there are only four possible arrangements, as shown below.

$P1$: $N_{G''}(v_1) = \{v_2, v_3, v_6\}$, $N_{G''}(v_2) = \{v_1, v_3, v_4\}$, $N_{G''}(v_4) = \{v_2, v_3, v_5\}$, $N_{G''}(v_5) = \{v_3, v_4, v_6\}$.
$P2$: $N_{G''}(v_1) = \{v_2, v_3, v_6\}$, $N_{G''}(v_2) = \{v_1, v_3, v_6\}$, $N_{G''}(v_4) = \{v_3, v_5, v_6\}$, $N_{G''}(v_5) = \{v_3, v_4, v_6\}$.
$P3$: $N_{G''}(v_1) = \{v_2, v_5, v_6\}$, $N_{G''}(v_2) = \{v_1, v_3, v_4\}$, $N_{G''}(v_4) = \{v_2, v_3, v_5\}$, $N_{G''}(v_5) = \{v_1, v_4, v_6\}$.
$P4$: $N_{G''}(v_1) = \{v_2, v_5, v_6\}$, $N_{G''}(v_2) = \{v_1, v_3, v_6\}$, $N_{G''}(v_4) = \{v_3, v_5, v_6\}$, $N_{G''}(v_5) = \{v_1, v_4, v_6\}$.

We find that $P1$ gives a Hamiltonian cycle $\{v_1, v_2, v_4, v_5, v_6, v_7, v_3, v_1\}$; $P3$ gives a Hamiltonian cycle $\{v_1, v_6, v_7, v_3, v_2, v_4, v_5, v_1\}$; $P4$ gives a Hamiltonian cycle $\{v_1, v_5, v_4, v_6, v_7, v_3, v_2, v_1\}$; $P2$ sets up a non-hamiltonian graph G''. Since $N_{G''}(v_1) = \{v_2, v_3, v_6\}$, we can conclude that $S_H = \{v_1, v_2, v_5\}$, $W_H = \{v_4\}$.

Thus, G'' can be written as $G'' = H_2 \vee (2K_2 \cup K_1)$; the two complete graphs in $2K_2$ are with $V(K_2) = \{v_i, v_{i+1}\}$ for $i = 1, 4$; and with $V(K_1) = \{v_7\}$. Since the number of components of $G'' - H_2$ is greater than $|H_2|$, by Theorem 3, G'' is not Hamiltonian. We can add two vertices x and y to G'' to obtain $G = H_4 \vee (2K_2 \cup K_1) = \eta_2$, and $\sigma_2(G) = n$. See Fig. 5.

Case 1.2.2 $l_d \leq n-4$.

Let $l_d = b$. We place the vertices of the Hamiltonian path $HP = \langle v_1, v_2, \ldots, v_a, v_{a+1}, \ldots, v_{l_d\text{-}1}, v_{l_d}, v_{l_d+1}, \ldots, v_{n\text{-}3}, v_{n\text{-}2}\rangle$ on the entries in the first row of a table, in which v_a is in the a^{th} column, v_{l_d} is in the b^{th} column, $v_{n\text{-}2}$ is in the $(n-2)^{\text{th}}$ column, and so on, where $a < b$. "S_H, T_H, W_H" corresponding to the Hamiltonian path HP are the entries of the 2^{nd} row. It can be seen that there are four possibilities for "S_H, T_H, W_H" to appear before the $(l_d)^{\text{th}}$ column, as shown in the following cases:

Case 1.2.2.1 If each entry from (2,1) to (2, $l_d - 1$), is S_H, then G'' has a Hamiltonian path $\langle v_1, v_2 = v_{l_1}, v_3 = v_{l_2}, \ldots, v_{d+1} = v_{l_d}, \ldots, v_{n\text{-}2}\rangle$. By Lemma 19 (3), G'' is not 2-connected.

Case 1.2.2.2 If there exist two consecutive entries, $(2, a) = T_H$ and $(2, a+1) = S_H$, before column l_d, then $v_a \in N_{G''}(v_{n\text{-}2})$ and $v_{a+2} \in N_{G''}(v_1)$. By a proper conversion, we can find a Hamiltonian Path $PT = \langle v_1, v_2, \nearrow, v_a, v_{n\text{-}2}, \searrow, v_{a+2}, v_{a+1}\rangle$. Note that v_{a+2} is located in the $(n-3)^{\text{th}}$ column. That means, in the Hamiltonian Path PT, the neighbor of v_1 that has the largest subscript is in the $(n-3)^{\text{th}}$ column, as shown in the third row of Table 3.

Specifically, rename the vertices in the Hamiltonian path PT such that $v_i' = v_i$ for $1 \leq i \leq a$ and $v_i' = v_{n+a-i-1}$ for $a + 1 \leq i \leq n-2$, as shown in the 4^{th} row of Table 3. Then we can find that $(v_1', v'_{n\text{-}3}) \in E$. It follows that the further discussion of this case is similar to that in Case 1.2.1.

Table 3. Hamiltonian Paths *HP* and *PT*.

		1	2	…	a	a+1	a+2	…	l_d –1	$b = l_d$	l_d +1	…	n–4	n–3	n–2
HP	1	v_1	v_2	⋯	v_a	v_{a+1}	v_{a+2}	⋯	$v_{l_d\text{-}1}$	v_{l_d}	v_{l_d+1}	⋯	$v_{n\text{-}4}$	$v_{n\text{-}3}$	$v_{n\text{-}2}$
	2	S_H		⋯	T_H	S_H	…	⋯	S_H	T_H		⋯	T_H	T_H	
PT	3	v_1	v_2	⋯	v_a	$v_{n\text{-}2}$	$v_{n\text{-}3}$	⋯	$v_{n+a\text{-}l_d}$	⋯	⋯	⋯	…	v_{a+2}	v_{a+1}
	4	v_1'	v_2'	⋯	v_a'	v_{a+1}'	v_{a+2}'	⋯	⋯	⋯	⋯	⋯	$v_{n\text{-}4}'$	$v_{n\text{-}3}'$	$v_{n\text{-}2}'$

Case 1.2.2.3 If there are three consecutive entries with $(2, w–1) = T_H$, $(2, w) = W_H$, $(2, w+1) = S_H$ before column l_d, then $v_{w\text{-}1} \in N_{G''}(v_{n\text{-}2})$, $v_w \notin N_{G''}(v_1)$, $v_w \notin N_{G''}(v_{n\text{-}2})$, $v_{w+1} \notin N_{G''}(v_1)$, $v_{w+1} \notin N_{G''}(v_{n\text{-}2})$, and $v_{w+2} \in N_{G''}(v_1)$. The Hamiltonian path $HP = \langle v_1, v_2, v_3, \ldots, v_{w\text{-}1}, v_w, v_{w+1}, \ldots, v_{n\text{-}4}, v_{n\text{-}3}, v_{n\text{-}2}\rangle$ is shown in the 1st row of Table 4. The sequence "S_H, T_H, W_H" corresponding to the Hamiltonian path HP are the entries of the 2nd row.

In this case, $deg_{G''}(v_1)+deg_{G''}(v_{n\text{-}2}) = (n – 4)$, if $deg_{G''}(v_1) > (n – 4)/2$, then $deg_{G''}(v_{n\text{-}2}) < (n – 4)/2$. Converting the Hamiltonian path HP to the following Hamiltonian path $PT1$: $PT1 = \langle v_1, v_2, \nearrow, v_{w\text{-}1}, v_{n\text{-}2}, \searrow, v_{w+1}, v_w\rangle$, as shown in the 3rd row of Table 4, we can see that $deg_{G''}(v_1) + deg_{G''}(v_w) = (n – 4)$, which implies that $deg_{G''}(v_w) < (n – 4)/2$. Hence, $deg_{G''}(v_{n\text{-}2}) + deg_{G''}(v_w) <(n – 4)$. This is a contradiction. Therefore, we must have $deg_{G''}(v_1) = deg_{G''}(v_{n\text{-}2})$ and n is even.

Subcase 1.2.2.3.1 $n \geq 12$.

Convert the Hamiltonian path HP to the Hamiltonian path $PT1$ as below:
$PT1 = \langle v_1, v_2, \nearrow, v_{w\text{-}1}, v_{n\text{-}2}, \searrow, v_{n\text{-}2\text{-}l_d+w}, \searrow, v_{l_d+2}, v_{l_d+1}, v_{l_d}, \searrow, v_{w+2}, v_{w+1}, v_w\rangle$. Note that v_{w+2} is located in the $(n–4)$th column. That means, in the hamiltonian Path $PT1$, the neighbor of v_1 that has the largest subscript is in the $(n–4)$th column, as shown in the third row of Table 4. The sequence "S_H, T_H, W_H" corresponding to the Hamiltonian path $PT1$ are the entries of the 4th row. Note that v_w is the end point of the Hamiltonian Path $PT1$. The $(4, w)$ entry is not S_H because $v_{n\text{-}3} \notin N_{G''}(v_1)$. The $(4, w)$ entry is also not T_H because $v_w \notin N_{G''}(v_{n\text{-}2})$. Hence, the $(4, w)$ entry must be W_H. In addition, the $(4, n+w–l_d–4)$ entry is not S_H because v_{l_d+1} is not a neighbor of v_1; the $(4, n+w–l_d–4)$ entry is not W_H because entry $(4, w) = W_H$; consequently, entry $(4, n+w–l_d–4) = T_H$. This shows that $(v_{l_d+2}, v_w) \in E$. Thus, $PT1$ can be converted to Hamiltonian path $PT2$ as below:
$PT2 = \langle v_1, v_2, \nearrow, v_{w\text{-}1}, v_{n\text{-}2}, \searrow, v_{n\text{-}2\text{-}l_d+w}, \searrow, v_{l_d+2}, v_w, v_{w+1}, \nearrow, v_{l_d}, v_{l_d+1}\rangle$. It can be seen that the neighbor of v_1 that has the largest subscript is in the $(n–3)$th column, as shown in the 5th row of Table 4. It follows that the further discussion of this case is similar to that in Case 1.2.1.

Table 4. Hamiltonian paths *HP*, *PT*1, and *PT*2.

		w–1	w	w+1	w+2	.	$b = l_d$	l_d+1	.	$n+w\text{-}l_d\text{-}4$	$n+w\text{-}l_d\text{-}3$	$n+w\text{-}l_d\text{-}2$	.	n–4	n–3	n–2
1	.	$v_{w\text{-}1}$	v_w	v_{w+1}	v_{w+2}	.	v_{l_d}	v_{l_d+1}	.	…	…	…	.	…	$v_{n\text{-}3}$	$v_{n\text{-}2}$
2	.	T_H	W_H	S_H	…	.	T_H	T_H						T_H	T_H	
*PT*1	.	$v_{w\text{-}1}$	$v_{n\text{-}2}$	$v_{n\text{-}3}$	…	.	$v_{n\text{-}2\text{-}l_d+w}$	…	.	v_{l_d+2}	v_{l_d+1}	v_{l_d}	.	v_{w+2}	v_{w+1}	v_w
4			W_H							T_H						
*PT*2	.	$v_{w\text{-}1}$	$v_{n\text{-}2}$	$v_{n\text{-}3}$	…	.	$v_{n\text{-}2\text{-}l_d+w}$	…	.	v_{l_d+2}	v_w	v_{w+1}	.	…	v_{l_d}	v_{l_d+1}

Subcase 1.2.2.3.2 $n = 10$.

On the left part of Table 5, the Hamiltonian path $HP = \langle v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\rangle$ and the sequence "S_H, T_H, W_H" are shown in the 1st and 2nd rows. From which we have $N_{G''}(v_1) = \{v_2, v_3, v_6\}$, and $N_{G''}(v_8) = \{v_7, v_3, v_6\}$. The Hamiltonian path $HP1 = \langle v_1, v_2, v_3, v_8, v_7, v_6, v_5, v_4\rangle$ obtained by converting HP and its sequence "S_H, T_H, W_H" are shown in the 4th and

Table 5. $\eta_1 = H_4 \vee 3K_2$.

	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
HP	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_1	v_2	v_3	v_4	v_5	v_6	v_8	v_7
2	S_H	S_H	T_H	W_H	S_H	T_H	T_H		S_H	S_H		W_H	S_H	T_H	T_H	
3	$N_{G''}(v_1) = \{v_2, v_3, v_6\}$, $N_{G''}(v_8) = \{v_3, v_6, v_7\}$								$(v_4, v_7)\notin E(G'')$, $(v_1, v_5) \notin E(G'')$, $(v_3, v_7)\in E(G'')$							
***HP*1**	v_1	v_2	v_3	v_8	v_7	v_6	v_5	v_4	v_2	v_1	v_3	v_4	v_5	v_6	v_7	v_8
5	S_H	S_H	T_H		S_H		T_H		S_H	S_H	T_H			T_H	T_H	
6	$(v_4, v_8)\notin E(G'')$&$(v_1, v_7)\notin E(G'')\Rightarrow(5,4)=W_H$ $deg_{G''}(v_4) = 3 \Rightarrow (v_4, v_6)\in E(G'')$&$(5,6)=T_H$								$(v_8, v_4)\notin E(G'')$&$(v_2, v_5)\notin E(G'')\Rightarrow(5,4)=W_H$; $deg_{G''}(v_2) = 3\Rightarrow(v_2, v_6)\in E(G'')$							
7	v_5	v_4	v_3	v_2	v_1	v_6	v_7	v_8								
8	S_H		T_H		S_H	T_H	T_H									
9	$(v_1, v_5)\notin E(G'')$&$(v_2, v_8)\notin E(G'')\Rightarrow(8,4)= W_H$; $deg_{G''}(v_5) = 3 \Rightarrow (v_3, v_5)\in E(G'')$&$(8,2) = S_H$															

5th rows. It can be seen that entries (5,1), (5,2), and (5,5) are all S_H; by $(v_4, v_3)\in E(G'')$ and $(v_4, v_5)\in E(G'')$, we have entry (5,3) = T_H and entry (5,7) = T_H; since $(v_4, v_8) \notin E(G'')$ and $(v_1, v_7)\notin E(G'')$, we have entry (5,4) = W_H; based on $deg_{G''}(v_4) = 3$, we can see that $(v_4, v_6) \in E(G'')$ and entry (5,6) = T_H. Thus $N_{G''}(v_4) = \{v_3, v_5, v_6\}$. Similarly, by proper conversions, we can find that $N_{G''}(v_5) = \{v_4, v_3, v_6\}$, $N_{G''}(v_7) = \{v_8, v_3, v_6\}$, and $N_{G''}(v_2) = \{v_1, v_3, v_6\}$ as shown in Table 5. Hence $G'' = H_2\vee(3K_2)$. Since the number of components of $G'' - H_2$ is greater than $|H_2|$, by Theorem 3, G'' is not Hamiltonian. By adding two vertices x and y to G'' such that $G = H_4\vee 3K_2$, we have $\delta(G) = 5$, $\sigma_2(G) = 10$, where η_1 is used to denote this kind of graph; that is $\eta_1 = H_4\vee 3K_2$. See Fig. 4.

Case 1.2.2.4 Only one entry in (2, 2) to (2, $l_d - 1$) is W_H and all others are S_H, as shown in Table 6.

Table 6. Only one W_H.

	1	2		w-1	w	w +1		l_d-1	$b = l_d$	l_d+1		l_d+(i-1)	l_d+i		n-3	n-2
1	v_1	v_2	⋯	$v_{w\text{-}1}$	v_w	v_{w+1}	…		v_{l_d}	v_{l_d+1}	…	…	…	…	$v_{n\text{-}3}$	$v_{n\text{-}2}$
2	S_H	S_H	…	S_H	W_H	S_H	…	S_H	T_H	T_H				…	T_H	
3	v_1	v_2	⋯	$v_{w\text{-}1}$	v_w	v_{w+1}	…		v_{l_d}	v_{l_d+1}	…	$v_{l_d+(i\text{-}1)}$	$v_{n\text{-}2}$	…	v_{l_d+i+1}	v_{l_d+i}
4	S_H	S_H	…	S_H	T_H	S_H	…	S_H						…	T_H	

Subcase 1.2.2.4.1

If there is an i such that $(v_w, v_{l_d+i}) \in E''$, where $i \in\{1, \ldots, n–3–l_d\}$, we can convert the Hamiltonian path whose vertices are shown in the first row of Table 6 to the following Hamiltonian path: $P_{l_d+i} = \langle v_1, v_2, \ldots, v_{l_d}, v_{l_d+1}, \ldots, v_{l_d+i\text{-}1}, v_{n\text{-}2}, \searrow, v_{l_d+i}\rangle$, as shown in the third row. Then the entry (4, w) will be T_H, as shown in the 4th row. This is a case belonging to Case 1.2.2.2.

Subcase 1.2.2.4.2

If there is no $(v_w, v_{l_d+i})\in E''$, where $i\in\{1, \ldots, n–3–l_d\}$, then, by the assumption that G'' is not Hamiltonian, none of $\{v_{l_d+1}, v_{l_d+2}, \ldots, v_{n\text{-}2}\}$ is adjacent to $\{v_1, v_2, \ldots, v_{l_d\text{-}1}\}$. Obviously, $\{v_{l_d}\}$ is one element vertex cut. Therefore, graph G'' is not 2-connected.

Case 2. G is not 1-*vertex-fault Hamiltonian*.

By Theorem 6 and $\kappa(G) \geq 4$, we have $G\subseteq \mathcal{G}_2$. According to Theorem 4, $\sigma_2(G') \geq n'-1$, G' is not Hamiltonian, and $\kappa(G') \geq 3$, we have $G' = H_{(n\text{-}2)/2} \vee \overline{K_{n/2}}$. Adding vertex x to G', we have $G = (H_{(n\text{-}2)/2}: x) \vee \overline{K_{n/2}} = H_{n/2} \vee \overline{K_{n/2}}$, $\delta(G) = n/2$, $\sigma_2(G) = n$, and $\kappa(G) \geq 4$. Note that $H_{n/2} \vee \overline{K_{n/2}}$ has been defined as η_8, which is isomorphic to $\mathcal{G}_2$. It is easy to see that $n \geq 8$ is required for ensuring $\delta(G) \geq 4$. See Fig. 12.

This completes the proof that either G is 2-vertex-fault Hamiltonian or $G \in \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8\}$. ❑

3. CONCLUDING REMARK

Following previous studies, we have completed the proof of the 2-vertex- fault-tolerance for graphs satisfying the degree conditions given by Ore. Since the 1-fault tolerance has been thoroughly studied, we further explore the 2-vertex-fault tolerance for any graph G with $\sigma_2(G) \geq n$ and $|G| = n$. This paper concludes that any G with $\sigma_2(G) \geq n$ and $\kappa(G) \geq 4$ must be 2-vertex-fault tolerant unless G belongs to one of the eight graph families. For a given graph G under the same conditions, other required conditions and other exceptional graph families for 2-edge-fault tolerance, or 1-vertex-1-edge-fault tolerance remain to be studied further.

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