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An Alternative Method for Cryptology in Secret Communication

MEHMET EMIR KOKSAL
Department of Mathematics
Ondokuz Mayis University
Samsun, 55139 Turkey
E-mail: mekoksal@omu.edu.tr

Commutativity of subsystems (SSs) in cascade connected forms to form larger systems gets worthy to improve noise disturbance, stability, robustness and many other properties in system design. There is a huge amount of work on the subject of commutativity of linear time-varying (LTV) systems and the mentioned improvements; these are referenced in the introduction. In this paper, another benefit of commutativity property is investigated in detail and illustrated by examples. This benefit is the gain of a new and original method for hiding the original (possibly secret) signals when sending them from one local area to another. Switching, pseudo-commutativity, and power-spectrum which are important for communication channels are extensively studied. It is shown that switching used for increasing safeness and slight deformations in commutativity conditions hardly spoil to attain the mentioned benefit. Hence, the paper presents an original and alternative method in cryptology. The results are all validated by illustrative examples and Matlab simulation toolbox Simulink.

***Keywords*:** commutativity, cryptology, communication, LTV system, security

1. INTRODUCTION

Second-order differential equations appear in many branches of engineering. They are used for a huge range of applications, including electrical systems, fluid systems, thermal systems and control systems. Especially, they are utilized as a powerful tool for modelling, analyzing, physical simulations and solving problems in modern system theory which is essential in any field of engineering and science.

In many cases, engineering systems are designed by interconnection of simple first or second-order systems to achieve beneficial properties such as easy controllability, design flexibility, less sensitivity to disturbances and robustness. Feedback and cascade connections are among the commonly used interconnection structures in control and communication systems, respectively. Cascade connection being an old but still an up to date design method [1-4] can be used to improve further different system performances in connection with the commutativity concept. Commutativity of traditional linear time-invariant systems is straightforward; however, LTV systems have found many applications recently [5-10]. Therefore, the subject of this paper is devoted on the commutativity of LTV systems only.

It is well-known that a cascade-connected system is a combination of two SSs so that the output of one is the input of the other [11]. If the input-output relation of the combination of two SSs in cascade form is not affected by the order of the connection

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then, these two systems are said to be commutative [12].

There is a great deal of literature about the commutativity of continuous LTV systems. Some of the important results about the commutativity are summarized in the sequel superficially.

J. E. Marshall is the first scientist studying on commutativity. In 1977, he proved that “for commutativity, either both systems are time-invariant or both systems are time-varying” [12]. Moreover, he proved necessary and sufficiently conditions for commutativity of first-order LTV systems. Then, investigations of commutativity conditions for second-order, third-order and fourth-order continuous LTV systems were studied in [13-17], respectively.

In 1988, M. Koksal introduced the basic fundamentals of the subject [18] which is the first and one of two tutorial exhaustive journal papers. The second work joint by the same author has presented explicit commutativity conditions of fifth-order systems in addition to reviews of commutativity of systems with non-zero initial conditions (ICs), commutativity and system disturbance, commutativity of Euler systems [19].

In [20], all the second-order commutative pairs of a first-order LTV analogue system are derived. In [21], the decomposition of a second-order LTV system into its first-order commutative pairs are studied. This is important for the cascade realization of second-order LTV systems.

Even though there is a large cycle of works on the commutativity of continuous-time systems, there is only one journal literature on the commutativity of discrete-time systems [22].

Some benefits of commutativity of LTV systems have already been appeared in the literature; for example, designing systems less sensitive to parameter values [23], reducing noise interference and disturbance [20], improving robustness [19].

The rest of paper is organized as focusing attention to the use of commutativity property in encrypting data transmission as follows: The next section constitutes the main idea of the paper as a new and original encrypting method using the commutativity property. Section 3 presents an example illustrating this original application. In the case of transmission using a single transmission channel (TC) which must be used in time-sharing for transmission paths $A \rightarrow B$ and $B \rightarrow A$ switching is necessary; switching and switching effects are investigated in Section 4. Further applications are possible by using nearly commutative (or pseudo-commutative) SSs and this is subjected in Section 5. For the same input-output signal pairs, the frequency spectrums of different transmitted signals (TSs) proceeded in the TC are compared in Section 6 to better illustrate the differences. Finally, the paper ends with Section 7 which includes conclusions.

2. NEW ENCRYPTING METHOD

This paper focuses attention on a new encrypting method of obscuring the information transmitted through any communication channel by disguising it between transmitter and receiver. More precisely, consider a communication system as shown in Fig. 1. In the figure, A and B represent commutative LTV systems so that both channels AB and BA produce the same output signal for any applied input signal. But the transferred signal from transmitter to receiver proceeds in completely different shapes through the transmitting medium. Hence, this generates somewhat prevention against the infiltrators to stealing the secret information during transmission.

More professional communication structure is indicated by using a single TC which is used by time-sharing between two channels of Fig. 1 is shown in Fig. 2.

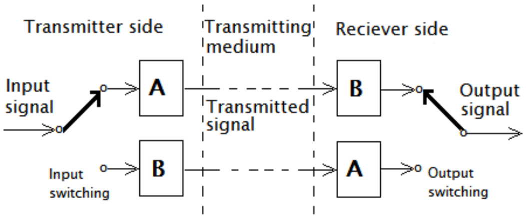

Fig. 1. Transmitting a secret input signal through a double TC.

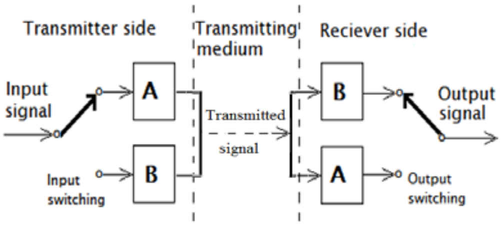

Fig. 2. Transmitting a secret input signal through a single TC.

The above concept can be extended to more complex structures by using higher number of switching greater than 1. For example, with two identical SSs A and two identical SSs B (commutative with A) 4 communication passages of the input signal can be achieved through transmitting medium to obtain the same output signal. In fact, the structures $A \to ABB$, $AA \to BB$, $AAB \to B$, $AB \to AB$ where the arrow "$\to$" separates SSs appearing in the transmitter and receiver sides. All these structures transfer any input signal to the same output signal which is transmitted in different shapes in transmitting medium of all four structures. The concept can be extended for more complicated cases by using more than two different commutative pairs.

3. ILLUSTRATIVE EXAMPLE

To see how any input signal is transmitted to the same output signal in different forms of the transmitting medium, consider the communication structure in Fig. 1 with the following example:

Example 1: Let the LTV SSs A and B be described by

$$A: \ddot{y}_A + \left(2 + 2sinw_0t\right)\dot{y}_A + \left(5 - \frac{1}{2}cos2w_0t + 2sinw_0t + w_0cosw_0t\right)y_A = x_A, \tag{1}$$

$$B = \frac{1}{2}\ddot{y}_B + \left(\frac{3}{4} + sinw_0t\right)\dot{y}_B + \left(\frac{409}{32} - \frac{1}{4}cos2w_0t + \frac{3}{4}sinw_0t + \frac{1}{2}w_0cosw_0t\right)y_B = x_B, \tag{2}$$

where x_i and y_i represent the input and output, respectively, of SSs $i = A, B$; (double) dot on the top indicates (second) time derivative.

This example due to the author is the first one appearing in the literature and it indicates the use of commutativity in cryptology [24] where some important aspects such as switching, pseudo-commutativity, power spectrum are not considered at all. This is not abnormal since the main aim of that paper is to study commutative pairs of well-known second-order differential systems. In this paper, we repeat this example in order to set up a base for the mentioned aspects and these subjects are considered in the sequel with this example and/or the others (Exs. 2, 3, 4). For example, power spectrum is studied using Ex. 1 as seen in Section 6.

It is straight forward to show that A and B are commutative since the time-varying coefficients of B can be obtained from those of A by the relation (3a) in [20]

$$\begin{bmatrix} b_2(t) \\ b_1(t) \\ b_0(t) \end{bmatrix} = \begin{bmatrix} a_2(t) & 0 & 0 \\ a_1(t) & a_2^{0,5}(t) & 0 \\ a_0(t) & f_A(t) & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}, \tag{3}$$

where $k_2 = 1/2$, $k_1 = -1/4$, $k_0 = 4213/400$ and $f_A = [2a_1 - (a_2)']/[4(a_2)^{0.5}] = 1 + \sin w_0 t$. Since $k_1 \neq 0$, the second one of the sufficient conditions of commutativity $A_0(t) = a_0 - (f_A)^2 - (a_2)^{0.5} f_A(t) = 3.5$ is satisfied for $A_0(t)$ being constant (see Eq. (2b) in [22]). It is easy to show that when the average values of coefficients are considered, both systems are asymptotically stable with eigenvalues $A_{1,2} = -1 \pm j2$, $B_{1,2} = -0.75 \pm j5$. This implies though not guaranties, the high possibility of stability of actual time-varying SSs A and B defined by Eqs. (1) and (2), respectively [24]; in fact, simulation results show that both systems are asymptotically stable.

To observe that both of the switching alternatives $A \to B$ and $B \to A$ shown in Fig. 1 where A and B are defined in Eqs. (1) and (2) with $w_0 = 2\pi$ yield the same output at the receiver side, an input signal ($30 \sin 1.2\pi t$ + *a saw-tooth of period* 3.3 s and increasing from -30 to $+30$) is applied on the transmitter side. As observed in Fig. 3, the transmitssions $A \to B$ and $B \to A$ yield the same output signal (see Output signal *10). In spite of the same input-output pairs for switching AB and BA, the travelled signals processed through transmission medium (TM) (see Transmitted s. A–B, Transmitted s. B–A) are quite different.

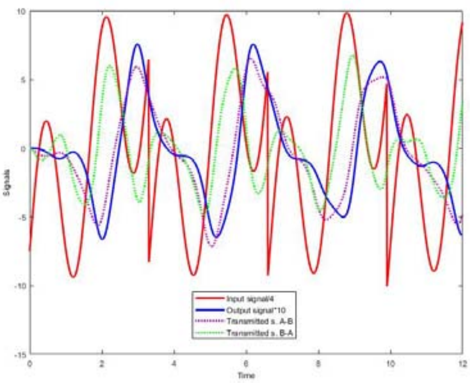

Fig. 3. Input, output and TSs in the communication system of Ex. 1.

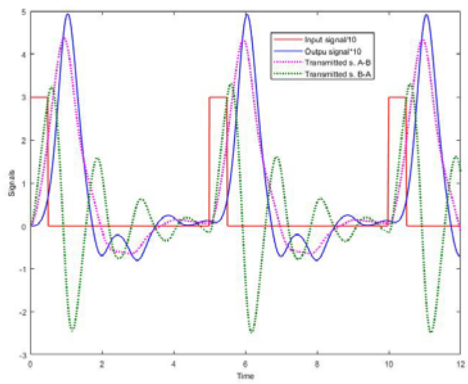

Fig. 4. Input, output and TSs in the communication system of Ex. 1 for a pulse train.

To verify that the discussions are independent of the input signal applied, the simulations are repeated with a pulse train of amplitude 30, period 5, and with a pulse width of 10 %. The input signal and the same output of both transmission switching paths $A \to B$ and $B \to A$ are shown in Fig. 4 (see Input signal/10, Output signal*10, respectively). It is also seen in this figure that the signals proceeded through the TM, namely (Transmitted s. *A–B*) and (Transmitted s. *B–A*), are quite different. Hence, the same output signal is received by channels *AB* and *BA* for the same input signal irrespective of the shape of the input signal whilst different signals are transmitted through the TM.

4. EFFECTS OF SWITCHING ON COMMUTATIVITY

When commutativity concept is used to transmit a signal through a TM secretly using a single TC time-shared by transmissions $A\rightarrow B$ and $B\rightarrow A$ as described in Section 3, a sufficiently high rate of switching is necessary to puzzle malicious persons or infiltrator to resolve the transmitted information. At the beginning of each transmission slots (say $A\rightarrow B$), some ICs had been formed in SSs *A* and *B* during the previous slot ($B\rightarrow A$ in this case) and these ICs may not satisfy the second commutativity condition for unrelaxed systems *A* and *B* at the initial time of the current time slot [6, Theorem 3.1 (Koksal 2)]. Therefore, the output of the switched transmission system will be different from those of non-switched systems *AB* and *BA*. This difference will be more appreciated if the damping properties of SSs *A* and *B* are weak (time constants are large with respect to switching period). This is because large time constants will elongate natural responses of SSs and the effect due to ICs that had been formed improperly for commutativity in the previous time slot. Hence, SSs used in single channel transmitting system described in Section 3 would better to have high damping coefficients for the output signal not affected by the switching considerably. This argument will be illustrated by the following two examples, namely Exs. 2 and 3:

Example 2: To illustrate the mentioned effect above, first consider SSs *A* and *B* which are commutative under zero ICs:

$$A: \dot{y}_A(t) + (1+\cos\pi t)y_A(t) = x_A(t);\, y_A(0) = 0, \tag{4}$$

$$B: \dot{y}_B(t) + (2+\cos\pi t)y_B(t) = x_B(t);\, y_B(0) = 0, \tag{5}$$

where *B* is obtained from *A* by using constants $c_1 = c_0 = 1$ (see Eq. (26) in [25]); namely,

$$\begin{bmatrix} b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1+cos\pi t & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1+cos\pi t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2+cos\pi t \end{bmatrix}.$$

Note the eigenvalues of *A* and *B* are $\lambda_A(t) = -1-\cos\pi t$ and $\lambda_B(t) = -2-\cos\pi t$, respectively. These eigenvalues remain in the left half of *s*-plane all the time (except the instants y = 1, 3, 5, … when it moves to origin instantly for subsystem *A*), hence both SSs are likely asymptotically stable [26].

For the commutativity of SSs under non-zero ICs as well, it is required by the above mentioned second commutativity condition (Theorem 3.1 in [19], Eq. (27) in [21]) that

$$c_1 + c_0 = 1, \tag{6}$$

$$y_A(t_s) = y_B(t_s), \tag{7}$$

where t_s is any switching instant (see Eqs. (11) and (12) in [25], respectively). When either one or two of these conditions are not satisfied, the systems AB and BA may not have the same output when excited by any input. In the present example, Eq. (6) is not obviously satisfied; further, there is no guaranty that Eq. (7) will be valid at the initial time of any switching slot since the ICs have been formed in the previous slot according to the dynamics of A and B rather independently. But, it is intuitively expected that the IC responses will decay fast to zero for highly damped SSs and the outputs of AB and BA will dominantly be determined by the forced response generated by the input signal. Hence, the coherence between the outputs AB and BA will not be affected considerably due to nonsatisfaction of the second commutativity condition spoiled by switching. SSs A and B defined by Eqs. (4) and (5), respectively, are examples of low damping systems (compared to SSs that will be considered in Ex. 3), so that switching is expected to will cause a great difference in the outputs when compared with the same output of AB and BA resulted without switching.

For an input $10\sin 2\pi t$ + *saw-tooth wave with period 3 magnitude* ±30, both systems AB and BA give the same output (Output signal*2.5) as shown in Fig. 5; in the same figure, the input signal, the TSs $A \to B$ and $B \to A$ are shown by (Input signal/2), (Transmitted s. $A-B$), Transmitted s. $B-A$), respectively.

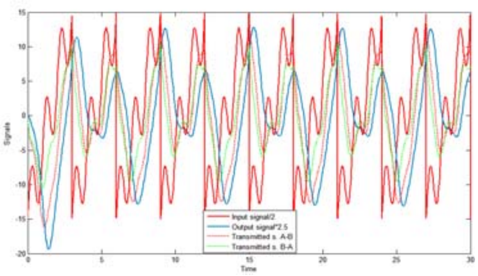

Fig. 5. Input, Output and TSs by paths $A \to B$ and $B \to A$ for Ex. 2.

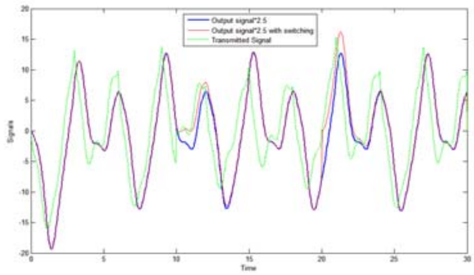

Fig. 6. TS in the single channel transmission and the output signals for Ex. 2.

To observe the effect of switching on the shape of the output signal, the paths $A \to B$ and $B \to A$ are switched periodically in sequence for durations of 10 seconds. The output signal at the receiver end is shown in Fig. 6 (Output signal *2.5 with switching); on the same figure, the output signal of connections AB and BA which appear in Fig. 5 (Output signal*2.5) is replotted. As it is expected, there is a difference between the direct communication with two lines without switching and communication by switching with a single transmission line; this difference is really apparent just after each switching instant for about 4-5 second duration and then disappears in the rest of the switching period so that the output coincides with the ideal case of direct communication without switching. This vacancy of switching can be reduced by using SSs having higher damping. Fig. 6 also includes the TS on the single line time shared by transmissions $A \to B$ and $B \to A$ (Transmitted s.).

Example 3: To observe the reduction of difference between two channel transmissions without switching and single channel transmission with switching, we consider similar SSs in Ex. 2 but having relatively higher damping then SSs of Ex. 2.

Consider the SSs A and B which are commutative under zero ICs:

$$A: \dot{y}_A(t) + (5 + \cos \pi t) y_A(t) = x_A(t); y_A(0) = 0,$$

$$B: \dot{y}_B(t) + (2 + \cos \pi t) y_B(t) = x_B(t); y_B(0) = 0.$$

Here, B is obtained from A by using constants $c_1 = 1$, $c_0 = -3$ (see Eq. (26) in [25]); namely,

$$\begin{bmatrix} b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 + cos\pi t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 + cos\pi t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 + cos\pi t \end{bmatrix}. \tag{8}$$

Note the eigenvalues of A and B are $\lambda_A(t) = -5 - \cos\pi t$ and $\lambda_B(t) = -2 - \cos\pi t$, respectively. These eigenvalues remain in the left half of s-plane all the time; in fact, exception of some instants for subsystem A in Ex. 2 does not occur in this case. Note also that the subsystem of Ex. 2 having larger damping or being more stable (subsystem B) is preserved in this example. Speaking about both SSs generally, those of Ex. 3 are more likely asymptotically stable and have higher damping than those of Ex. 2 [26]. Therefore, due to the reasons explained in Ex. 2, using a single channel by switching is expected to cause less deviation in the outputs when compared with the same output of AB and BA resulted without switching.

For the same input as in Ex. 2, both systems AB and BA give the same output (Output signal*2.5) as shown in Fig. 7; on the same figure, the input signal, the TSs $A \to B$ and $B \to A$ are shown by (Input signal/2), (Transmitted s. A–B), (Transmitted s. B–A), respectively.

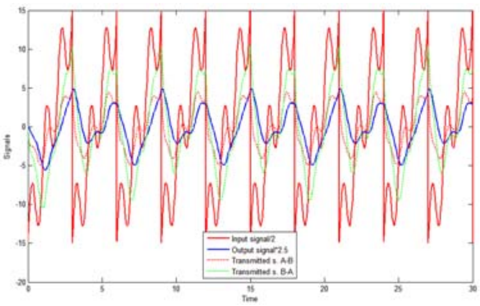

Fig. 7. Input, Output and TSs by paths $A \to B$ and $B \to A$ for Ex. 3.

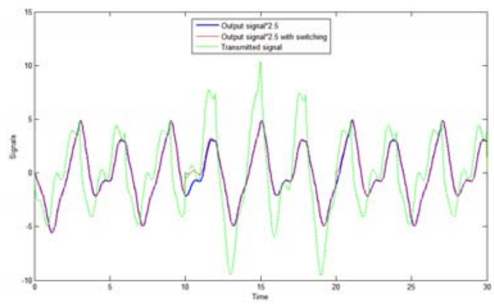

Fig. 8. TS in the single channel transmission and the output signals for Ex. 3.

To observe the effect of switching on the shape of the output signal, the paths $A \to B$ and $B \to A$ are switched periodically in sequence for durations of 10 seconds. The output signal at the receiver end is shown in Fig. 8 (Output signal*2.5 with switching); on the same figure, the output signal of connections AB and BA which appear in Fig. 5 (Output signal*2.5) is replotted. As it is expected, there is a difference between the direct communication with two lines without switching and communication by switching with a single transmission line; this difference is really apparently just after each switching instant for about smaller than 1.5 second duration and then disappears in the rest of the switching period so that the output coincides with the ideal case of direct communication without switching. Note that this vacancy of switching is reduced from 5-7 of Ex. 2 to values less than 1.5 s by using SSs having higher damping in Ex. 3. Fig. 8 also includes the TS on the single line time-shared by transmissions $A\rightarrow B$ and $B\rightarrow A$ (Transmitted signal). As a conclusion, effects of switching between the TCs can be better reduced by using highly damped SSs A, B.

5. USE OF PSEUDO-COMMUTATIVE SUBSYSTEMS

In the following example, how the use of commutativity in cryptology can be expanded by using the concept of nearly-commutative SSs. Since the purpose is just to introduce this goal, first-order SSs are considered for simplicity.

Example 4: Let A be the first-order LTV system of Ex. 3, namely defined by

$$A: \dot{y}_A(t) + (5 + \cos \pi t) y_A(t) = x_A(t); y_A(0) = 0.$$

The system is chosen purposely as to have highly damped characteristic value always remaining in the left half of s-plane far away from the imaginary axis. This is because to have sufficiently fast decaying natural responses due to mismatching ICs preventing commutativity as mentioned before. Otherwise, when switching takes place as described in Section 4, nonzero ICs not satisfying the commutativity conditions that had been formed before the switching from $A\rightarrow B$ to $B\rightarrow A$ or vice versa will occur and these paths will not give the same responses. We now consider the transformation in Eq. (8) which gives all the commutative pairs of Subsystem A; instead of choosing c_1 and c_2 as constants, let us choose them as parameters varying slowly with respect to the natural dynamics of SSs A and B. This is expected to result with nearly commutative SSs so that the cascade connections AB and BA will yield almost the same outputs whilst the spectrum of the TS trough the communication medium continuously changing with varying parameters c_1 and c_2; this will puzzle the third persons trying to catch the actual information illegally. To observe this, we choose $c_1 = 2 + \sin 0.1\pi t$ and $c_0 = -3\cos 0.2\pi t$ in the mention respect and by a similar equation to Eq. (8) we obtain Subsystem

$$B: (2 + \sin 0.1\pi t)\dot{y}_B(t) + (105 \sin 0.1\pi t + 3\cos 0.2\pi t) y_B(t) = x_B(t); y_B(0) = 0.$$

For an input $x(t) = 12\sin 2\pi t$ which is shown in Fig. 9 (Input/10), the outputs of transmissions $A\rightarrow B$ and $B\rightarrow A$ are also plotted in the figure (Output*50: *AB* and Output*50: *BA*, respectively). Even though A and B are not exactly commutative. It is seen that AB and BA almost produce the same output owing to the slow variations of parameters c_1 and c_0 used to obtain A from B through a similar equation to Eq. (8). Further, on the same figure is shown the output (Output*50: *AB–BA* switched.) of the single line system which is used in time-sharing between transmissions $A\rightarrow B$ and $B\rightarrow A$. It is obvious that switching used for a single channel transmission does not spoil to achieve the same response of systems AB and BA.

In Fig. 10, it is seen that although all outputs are almost the same, the shape of the proceeding signals on the channel $A\rightarrow B$ (Transmitted s. *AB*), on the channel $B\rightarrow A$ (Transmitted s. *BA*), and on the common channel (Transmitted s. *AB–BA* switched) are quite different. Hence, the same output signal is transferred through the TM in different forms and this complicates attaining it by unauthorized persons.

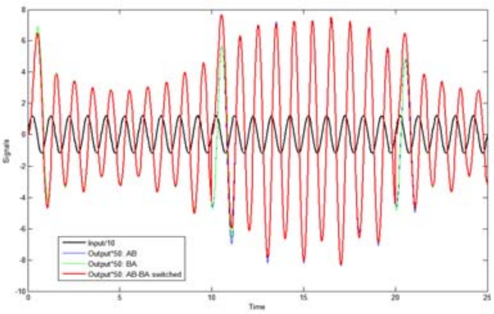

Fig. 9. Output signals obtained at the receiver side for Ex. 4.

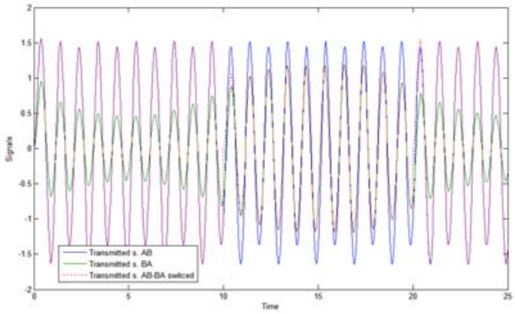

Fig. 10. TSs travelling through TM for Ex. 4.

6. POWER SPECTRUMS

On the base of their frequency spectrums, this section explains the use of commutativity for encrypting signals when travelling through the TCs. Since the power or frequency spectrum is an important subject for comparing signals and is a very essential analysis tool in communication theory, especially for studying different modulation techniques, the comparison of signals transmitted through the TM depicted in Fig. 1 by comparing their spectrums is essential and this is considered in this section.

For Example 1, the spectrums of the TS from Subsystem *A* to Subsystem *B* and the TS from Subsystem *B* to Subsystem *A* are shown in Fig. 11 by (Transmitted *AB*) and (Transmitted *BA*), respectively. Obviously, the spectrums are quite different in spite of the fact these signals produce the same outputs at the receiver end (see Output signal*10 in Fig. 3).

It has been already noted that the distortive effects of using switching for single channel communication as described in Section 4 and pseudo-commutative SSs to strength cryptologic actions as illustrated in Section 5 are hardly seen at the receiver output, the propagating signals from transmitter to receiver side depicted in Figs. 5, 7, and 10 will naturally contain quite different spectrums similar to those in Fig. 11; whilst they are producing almost the same output signal on the receiver side (See Figs. 6, 8, and 9, respectively). Therefore, it is satisfied with this much dealing about the frequency spectrums characteristics.

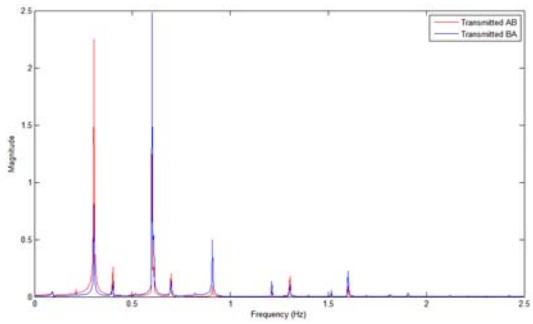

Fig. 11. Power spectrums of the signals transmitted $A \rightarrow B$ and $B \rightarrow A$ for Ex. 1.

7. CONCLUSIONS

Cryptology is an important subject for hiding signals in communication systems transferring information from one local area to another. In this paper, how commutativity property of SSs in a communication system can be used for transmitting signals safely by reducing the probability of stealing by unauthorized persons. In fact, it is shown that the same output signal at the receiver side of a communication channel can be transmitted simultaneously through the same channel by using commutative SSs at the transmitter and receiver sides together with switching while changing its transmitted version through the TM.

Instead of fixing some system parameters as in c_1 and c_0 in Ex. 4, some certain time-change for c_1 and c_0, for example changing them arbitrarily but slowly with time, will yield using pseudo-commutative SSs and thus additional alternatives for hiding the transmitted information.

Moreover, the TS trough the TM in case of communication on a signal channel can be further puzzled by changing the switching strategy; for example, changing the switching frequency and switching periods of channels *AB* and *BA* unsymmetrically will produce extra advantages for a safe communication. And this can be forwarded as a further research subject on the area.

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Mehmet Emir Koksal received his Ph.D. degree on Mathematics from Gebze Institute of Technology, Turkey in 2009. After he received his Ph.D., he studied as a postdoctoral scholar for one year at the Department of Mathematical Sciences of Florida Institute of Technology, Melbourne. His research interests include development of numerical solution methods for PDEs using finite difference, element and volume methods, mathematical modeling and analyzing of various engineering problems using PDEs, commutativity of continuous and discrete time-varying linear systems and their applications.