JOURNAL OF INFORMATION SCIENCE AND ENGINEERING 36, 1191-1210 (2020)
DOI: 10.6688/JISE.202011_36(6).0004

# Estimation for Exponentiated Weibull Distribution under Accelerated Multiple Type-II Censored Samples*

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Products or materials under modern technology has a long period of time, and the problem of obtained sufficient information in small period of time need to stress higher than normal conditions. In this article, we address the problem of partially constant-stress accelerated life tests (ALTs) with multiple Type-II censored scheme to estimate exponentiated Weibull (EW) life time distribution. The maximum likelihood and Bayes estimators of the distribution parameters and acceleration factor are developed. Also, the credible and approximate confidence intervals of the parameters are discussed. Although, the Bayes estimators cannot obtained in a plain form, then Markov chain Monte Carlo (MCMC) methods is carried out to draw samples from the posterior distribution. Finally, the estimation procedures are compared and assessed for the unknown parameters inspected over numerical discussions.

***Keywords:*** multiple Type-II censoring, accelerated life tests, exponentiated Weibull distribution, maximum likelihood estimation, Bayes MCMC estimation

## 1. INTRODUCTION

Accelerated life testing (ALTs) could be applied to obtain quick information about the lifetimes products. In ALTs, the tested units is conducted at higher stress than normal stress levels (say pressure, temperature, voltage, *etc.*) to encourage early failures. At such accelerated conditions the collected data are investigated via a physically appropriate statistical model and is used in estimation problem of the life times distribution at normal stress conditions. [1, 2] introduced the concept of ALT, through this case, partially ALTs would be a perfect nominee to proceed the life test in order to the main presumption in partially ALT are the stress is not known and the acceleration factor can't be presupposed. In partially ALTs, units is tested at the two case, normal and stress conditions [3]. The partially ALTs is particulary useful in new-product development. However, an appropriate statistical model are extrapolated to the data collected.

ALTs and partially ALT can be carried out by three common methods of ALTss: step-stress, constant-stress and progressive-stress. The main difference between the three

Received December 17, 2019; revised December 31, 2019; accepted January 8, 2020.
Communicated by Mahmoud Abdel-Aty.
* This work was supported by the Deanship of Scientific Research at Umm Al-Qura University at grant code: (19-SCI-1-03-0009).

methods of ALTs is defined by the relation between testing time and the stress loading. In constant-stress ALTs, the stress kept constant during the life of test products. In step-stress ALTs, each tested unit starts in a particular low level of stress, if the unit does not fail at a particular time, then the level of stress is change to a higher level. A simple step-stress ALTs employ only two levels of stress [4-7]. Progressive-stress ALTs scheme lets the stress on a test unit to be increased continuously, [8]. There are two types of partial ALTs: partial step-stress ALTs and partial constant-stress ALTs [9-11]. In this study, partial constant-stress ALTs schemes are used.

In the frame of partial constant-stress ALTs, the maximum likelihood (ML) method is applied for estimates the model parameters for compound Rayleigh distribution [12], for generalized exponential distribution [13], Burr XII distribution [14], Rayleigh distribution [15] and Weibull distribution [16]. The problem on optimal design of partial constant-stress ALTss is discussed with [17, 18] for truncated logistic distribution under conditions of decreasing of maximum likelihood estimators the generalized asymptotic variance for the model parameters. Also, the estimation of model parameters of inverted Weibull distribution under partial constant-stress ALTs see [19].

The censoring technique is very popular and familiar in a life tests. A common ones in life testing are Type-I and Type-II censoring which can't authorize the experimenter to withdraw units from the life testing at several stages through the experiment other the final point, and Type-II Hybrid as a joint of Type-I and Type-II of Type-II see [20]. But progressive censoring permit for items to be extracted through control conditions see [21]. This problem can be overcome by applying more general censoring scheme CS, called multiple Type-II censored scheme see [22]. However, multiple Type-II censoring schemes will be well choice in this situation it authorizes for units to be extracted at any stage through the life test period. Multiple Type-II censoring can be occurred in the test that units fails with more than one cause. Hence the two type, Type-I and Type-II are considered as a special case of numerous censoring. For a survey on the scheme, one may refer to [23, 24].

Exponentiated Weibull (EW) distribution, from the most popular models introduced as an expansion distribution of the renowned Weibull distribution [25, 26]. The EW family includes distribution has been extensively used for analyses the life time data aside from a broader class with some monotone failure ratios. The various moments with no restrictions imposed on the model presented in closed-form in [27]. The EW distribution with two parameters $\alpha$ and $\theta$ has the probability density function (PDF) defined by

$$f_1(t) = \alpha\theta t^{\alpha-1}\exp[-t^{\alpha}](1-\exp[-t^{\alpha}])^{\theta-1},\ \alpha,\ \theta > 0,\ t > 0, \tag{1}$$

and the cumulative distribution function (CDF), hazard rate function (HRF) and the reliability function (RF) defined by

$$F_1(t) = (1-\exp[-t^{\alpha}])^{\theta}, \tag{2}$$

$$H_1(t) = \frac{\alpha\theta t^{\alpha-1}\exp[-t^{\alpha}](1-\exp[-t^{\alpha}])^{\theta-1}}{1-(1-\exp[-t^{\alpha}])^{\theta}}, \tag{3}$$

and

$$S_1(t) = 1-(1-\exp[-t^{\alpha}])^{\theta}. \tag{4}$$

This article is planned to take the following stages, the model formulation and it's assumptions are derived in Section 2. The Maximum likelihood estimators (MLEs) of the unknown parameters as well as the approximate confidence intervals are discussed in Section 3. Also, Section 4, the Bayes approach is applied with the help of MCMC method likewise the credible intervals for the parameters are presented. Numerical measurements for the theoretical results discussed in the two numerical example and simulation study in Section 5. Finally, it is finished with several Brief comments.

## 2. MODEL DESCRIPTION

In the model of partially constant-stress ALTss, the total tested units $n$ are split randomly to two groups, the group 1 has $n_1$ units tested under normal condition and the group 2 contains $n_2 = n - n_1$ units tested under stress condition. Multiple Type-II censored is utilized as follows. Suppose the two group $n_j,\ j = 1,2$ are gathered in a life testing and the prior of the experiment fixed integer $m_j$ is determined so that the $r_{j1}th, r_{j2}th....r_{jmj}$ th failure times are only available, that is $T_{r_{j1};nj} < T_{r_{j2};nj} < ... < T_{r_{jm_j};nj}, j = 1,2$ is called multiple Type-II censored data, where $r_{j1} + r_{j2} + ... + r_{jm_j} = m_{j1}$. If $n_j$ failure times has distribution with CDF given by $F_j(x)$ and PDF given by $f_j(x)$, then the joint density of observed multiply Type-II censored data $\underline{t} = t_{r_{j1};nj} < t_{r_{j2};nj} < ... < t_{r_{jm_j};nj}$ and for simplicity take $t_{r_{ji};nj} = t_{r_{ji}}$ (see Kang and Park [22]) is given by

$$L(\Theta|\underline{t}) = \prod_{j=1}^{2} n_j! \prod_{i=1}^{m_j} f_j(t_{r_{ji}}) \prod_{i=1}^{m_j+1} \frac{[F_j(t_{r_{ji}}) - F_j(T_{r_{j(i-1)}})]^{r_{ji}-r_{j(i-1)}-1}}{(r_{ji} - r_{j(i-1)} - 1)!}, \tag{5}$$

where $\Theta = (\alpha, \theta, \beta)$.

**The basic assumptions are:**

(i) The lifetime of any unit tested at normal stress condition are i.i.d random variables with EW lifetime distribution with parameters $\alpha$ and $\theta$ given by Eq. (1).

(ii) To handle effects of environment or stresses on the lifetime distribution, the parametric proportional hazard model (also named Cox model) can be applied. Thus, $h_2(t) = \beta h_1(t)$ and then the survival function under accelerated condition is given by, $S_2(t) = (S_1(t))^{\beta}$, where beta is an acceleration factor satisfying $\beta > 1$. Also, the accelerated lifetime model, in which assuming that $S_2(t) = S_1(\beta t)$, Both model can be applied, but we consider the first model in this paper. Then, the HRF, RF, CDF and PDF under parametric proportional hazard model are presented, as follows:

$$H_2(t) = \frac{\beta \alpha \theta t^{\alpha-1} \exp[-t^{\alpha}](1 - \exp[-t^{\alpha}])^{\theta-1}}{1 - (1 - \exp[-t^{\alpha}])^{\theta}}, \tag{6}$$

$$S_2(t) = (1 - (1 - \exp[-t^{\alpha}])^{\theta})^{\beta}, \tag{7}$$

$$F_2(t) = 1 - (1 - (1 - \exp[-t^{\alpha}])^{\theta})^{\beta}, \tag{8}$$

and

$$f_2(t) = \beta\alpha\theta t^{\alpha-1}\exp[-t^{\alpha}](1-\exp[-t^{\alpha}])^{\theta-1}(1-(1-\exp[-t^{\alpha}])^{\theta})^{\beta-1}$$
$$,t>0,\ \alpha,\ \theta>0,\ \beta>1. \quad (9)$$

(iii) The lifetimes of items $T_{r_{ji}}, j=1,2, i=1,...,n_j$, of units assigned to use condition $(j=1)$ and accelerated condition $(j=2)$ are mutually independent and identically distributed *iid* random variables.

## 3. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we discussed the parameters estimation with one of important classical methods, say ML method, see [28-30]. The joint likelihood function of the model parameters $\alpha$, $\theta$ and $\beta$ of the two EW distributions given by Eqs. (1), (2), (8) and (9) with the joint multiple Type-II censored sample $\underline{T}=\left\{T_{r_{j1}},T_{r_{j2}},...,T_{r_{jm_j}}\right\}, j=1,2$ without normalized constant, as follows

$$\begin{aligned}L(\alpha,\theta,\beta|\underline{t}) =&(\alpha\theta)^{m_1+m_2}\beta^{m_2}\exp\left\{(\alpha-1)\left\{\sum_{i=1}^{m_1}\log t_{r_{1i}}+\sum_{i=1}^{m_2}\log t_{r_{2i}}\right\}-\sum_{i=1}^{m_1}t_{r_{1i}}^{\alpha}\right.\\&-\sum_{i=1}^{m2}t_{r_{2i}}^{\alpha}+(\theta-1)\left\{\sum_{i=1}^{m_1}\log U_{r_{1i}}+\sum_{i=1}^{m2}\log U_{r_{2i}}\right\}+(\beta-1)\\&\times\sum_{i=1}^{m2}\log\left(1-U_{r_{2i}}^{\theta}\right)+\sum_{i=1}^{m_1+1}(r_{1i}-r_{1(i-1)}-1)\log\left[U_{r_{1i}}^{\theta}-U_{r_{1(i-1)}}^{\theta}\right]\\&\left.+\sum_{i=1}^{m_2+1}(r_{2i}-r_{2(i-1)}-1)\log\left[\left(1-U_{r_{2(i-1)}}^{\theta}\right)^{\beta}-\left(1-U_{r_{2i}}^{\theta}\right)^{\beta}\right]\right\}. \end{aligned} \quad (10)$$

The log-likelihood function $\ell(\alpha,\theta,\beta|\underline{t})=logL(\alpha,\theta,\beta|\underline{t})$ is then presented by

$$\begin{aligned}\ell(\alpha,\theta,\beta|\underline{t}) =&(m_1+m_2)\log\alpha\theta+m_2\log\beta+(\alpha-1)\left\{\sum_{i=1}^{m_1}\log t_{r_{1i}}+\sum_{i=1}^{m_2}\log t_{r_{2i}}\right\}\\&-\sum_{i=1}^{m_1}t_{r_{1i}}^{\alpha}-\sum_{i=1}^{m_2}t_{r_{2i}}^{\alpha}+(\theta-1)\left\{\sum_{i=1}^{m_1}\log U_{r_{1i}}+\sum_{i=1}^{m_2}\log U_{r_{2i}}\right\}\\&+(\beta-1)\sum_{i=1}^{m_2}\log(1-U_{r_{2i}}^{\theta})+\sum_{i=1}^{m_1+1}(r_{1i}-r_{1(i-1)}-1)\log[U_{r_{1i}}^{\theta}-U_{r_{(i-1)}}^{\theta}]\\&+\sum_{i=1}^{m_2+1}(r_{2i}-r_{2(i-1)}-1)\log[(1-U_{r_{2(i-1)}}^{\theta})^{\beta}-(1-U_{r_{2i}}^{\theta})^{\beta}],\end{aligned} \quad (11)$$

where

$$U_{r_{ji}}=1-exp[-t_{r_{ji}}^{\alpha}], \quad (12)$$

and

$$\frac{\partial U_{r_{ji}}}{\partial \alpha} = \alpha t_{r_{ji}}^{\alpha-1} exp[-t_{r_{ji}}^{\alpha}] = \grave{U}_{r_{ji}}. \tag{13}$$

The ML estimators of the model parameters $\alpha$, $\beta$ and $\theta$ are determined as follows:

$$\begin{aligned}\frac{\partial \ell(\alpha,\theta,\beta|\underline{t})}{\partial \beta} =& \frac{m_2}{\beta} + \sum_{i=1}^{m_2} \log(1-U_{r_{2i}}^{\theta}) + \sum_{i=1}^{m_2+1} (r_{2i} - r_{2(i-1)} - 1) \\ &\times \frac{(1-U_{r_{2(i-1)}}^{\theta})^{\beta} \log(1-U_{r_{2(i-1)}}^{\theta}) - (1-U_{r_{2i}}^{\theta})^{\beta} \log(1-U_{r_{2i}}^{\theta})}{(1-U_{r_{2(i-1)}}^{\theta})^{\beta} - (1-U_{r_{2i}}^{\theta})^{\beta}} = 0, \end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial \ell(\alpha,\theta,\beta|\underline{t})}{\partial \theta} =& \frac{m_1+m_2}{\theta} + \sum_{i=1}^{m_1} \log U_{r_{1i}} + \sum_{i=1}^{m_2} \log U_{r_{2i}} + (\beta-1) \sum_{i=1}^{m_2} \frac{U_{r_{2i}}^{\theta} \log U_{r_{2i}}}{1-U_{r_{2i}}^{\theta}} \\ &+ \sum_{i=1}^{m_1+1} (r_{1i} - r_{1(i-1)} - 1) \frac{U_{r_{1i}}^{\theta} \log U_{r_{1i}} - U_{(r_{1(i-1)})}^{\theta} \log U_{r_{1(i-1)}}}{U_{r_{1i}}^{\theta} - U_{r_{1(i-1)}}^{\theta}} \\ &-\beta \sum_{i=1}^{m_2+1} (r_{2i} - r_{2(i-1)} - 1) \left\{ \frac{(1-U_{r_{2(i-1)}}^{\theta})^{\beta-1} U_{r_{2i}}^{\theta} \log U_{r_{2(i-1)}}}{(1-U_{r_{2(i-1)}}^{\theta})^{\beta} - (1-U_{r_{2i}}^{\theta})^{\beta}} \right. \\ &\left. - \frac{(1-U_{r_{2i}}^{\theta-1})^{\beta-1} U_{r_{2i}}^{\theta} \log U_{r_{2i}}}{(1-U_{r_{2(i-1)}}^{\theta})^{\beta} - (1-U_{r_{2i}}^{\theta})^{\beta}} \right\} = 0, \end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial \ell(\alpha,\theta,\beta|\underline{t})}{\partial \alpha} =& \frac{m_1+m_2}{\alpha} + \sum_{i=1}^{m_1} \log t_{r_{1i}} + \sum_{i=1}^{m_2} \log t_{r_{2i}} - \Sigma_{i=1}^{m_1} t_{r_{1i}}^{\alpha} \log t_{r_{1i}} - \sum_{i=1}^{m_2} t_{r_{2i}}^{\alpha} \log t_{r_{2i}} \\ &-\theta(\beta-1) \sum_{i=1}^{m_2} \frac{U_{r_{2i}}^{\theta-1} \acute{U}_{r_{2i}}}{1-U_{r_{2i}}^{\theta}} + \theta \sum_{i=1}^{m_1} (r_{1i} - r_{1(i-1)} - 1) \\ &\times \frac{U_{r_{1i}}^{\theta-1} \acute{U}_{r_{1i}} - U_{r_{1(i-1)}}^{\theta-1} \grave{U}_{r_{1(i-1)}}}{U_{r_{1i}}^{\theta} - U_{r_{1(i-1)}}^{\theta}} - \beta \sum_{i=1}^{m_2} (r_{2i} - r_{2(i-1)} - 1) \\ &\times \frac{(1-U_{r_{2(i-1)}}^{\theta})^{\beta-1} U_{r_{2(i-1)}}^{\theta-1} \grave{U}_{r_{2(i-1)}} - (1-U_{r_{2i}}^{\theta})^{\beta-1} U_{r_{2i}}^{\theta} \grave{U}_{r_{2i}}}{(1-U_{r_{2(i-1)}}^{\theta})^{\beta} - (1-U_{r_{2i}}^{\theta})^{\beta}} = 0. \end{aligned} \tag{16}$$

Since the closed form for the solutions of nonlinear Eqs. (14)-(16) are difficult to obtain, the numerical method, quasi-Newton Raphson algorithm, is applied for these simultaneous equations for obtaining MLEs of $\alpha$, $\beta$ and $\theta$.

### 3.1 Asymptotic Confidence Intervals CIs

Since the MLEs of $\alpha$, $\theta$ and $\beta$ can't be acquired in closed formula, then we can't deduce the exact distribution of the MLEs. For EW distribution in partially constant-stress ALTs, the MLEs of $\alpha$, $\theta$ and $\beta$ has an asymptotic variance-covariance matrix determine by inverting the Fisher information matrix, which substituting $\hat{\alpha}$ for $\alpha$, $\hat{\theta}$ for $\theta$ and $\hat{\beta}$ for $\beta$. The Fisher information matrix of estimators $\hat{\alpha}$; $\hat{\theta}$ and $\hat{\beta}$ is presented by calculating the second partial derivative respected to the log-likelihood function. The information matrix can be utilized to build the variance-covariance matrix for the tree parameters.

Using the asymptotic distributions of the elements for the vector $\varphi(\alpha,\theta,\beta)$, the approximate CIs of the parameters is obtained by

$$((\hat{\alpha}-\alpha),(\hat{\theta}-\theta),(\hat{\beta}-\beta)) \sim N(0, I^{-1}(\hat{\alpha},\hat{\theta},\hat{\beta})), \tag{17}$$

while $I^{-1}(\hat{\alpha},\hat{\theta},\hat{\beta})$ is estimate at the MLEs ($\hat{\alpha}$, $\hat{\theta}$ and $\hat{\beta}$) from the negative values of the second partial derivatives of the log-likelihood function. The performance of $I_{ij}(\alpha,\theta,\beta), i,j = 1,2,3$, which is written by $I_{ij}(\hat{\alpha},\hat{\theta},\hat{\beta})$, where

$$I_{ij}(\varphi) = -\frac{\partial^2 \ell(\varphi|\underline{y})}{\partial\varphi_i \partial\varphi_j}, \tag{18}$$

$$I_0^{-1}(\hat{\alpha},\hat{\theta},\hat{\beta}) = \begin{bmatrix} -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\alpha 2} & -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\alpha\partial\theta} & -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\alpha\partial\beta} \\ -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\theta\partial\alpha} & -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\theta 2} & -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\theta\partial\beta} \\ -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\beta\partial\theta} & -\frac{\partial^2\ell(\alpha,\theta,\beta|\underline{t})}{\partial\beta 2} \end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\theta},\hat{\beta})}. \tag{19}$$

So, the 100(1-$\gamma$)% approximate CIs of $\alpha,\theta$, and $\beta$ are then obtained as

$$\hat{\alpha} \mp z_{\frac{\gamma}{2}}\sqrt{v_{11}},\ \hat{\theta} \mp z_{\frac{\gamma}{2}}\sqrt{v_{22}} \text{ and } \hat{\beta} \mp z_{\frac{\gamma}{2}}\sqrt{v_{33}}, \tag{20}$$

where, the elements on the diagonal of the matrix $I^{-1}(\hat{\alpha},\hat{\theta},\hat{\beta})$ are $v_{11}$, $v_{22}$ and $v_{33}$. Also, $z_{\frac{\gamma}{2}}$ is standard normal values computed with right-tail probability $\frac{\gamma}{2}$.

## 4. BAYES ESTIMATION

In this section, we focuses on the Bayesian approach for estimating the unknown model parameters $\alpha$, $\beta$ and $\theta$ using square error (SE) and general entropy (GE) loss function. Here, the desired Bayesian estimates under the stated loss functions (squared error and entropy) are acquired which are defined respectively as;

$$L_{SE}(\hat{\varphi}(\theta),\varphi(\theta)) = (\hat{\varphi}(\theta)-\varphi(\theta))^2,$$
$$L_{GE}(\hat{\varphi}(\theta),\varphi(\theta)) \propto \left(\frac{\hat{\varphi}(\theta)}{\varphi(\theta)}\right)^w - w\log\left(\frac{\hat{\varphi}(\theta)}{\varphi(\theta)}\right) - 1, w \neq 0.$$

The estimate of $\varphi(\theta)$ is $\hat{\varphi}(\theta)$. Bayesian estimator together with the assumption that $\alpha$ and $\theta$ are independent, where $\alpha$ and $\theta$ have the following gamma prior density distribution, respectively

$$\pi(\alpha) \propto \alpha^{a-1}\exp(-b\alpha) \text{ and } \pi(\theta) \propto \theta^{c-1}\exp(-d\theta), a,b,c,d > 0, \tag{21}$$

where $a$, $b$, $c$ and $d$ are known shape and scale parameters. Also, for the accelerate factor $\beta$ the following non-informative prior (NIP) is given as:

$$\pi(\beta) \propto \beta^{-1}, \beta > 1. \tag{22}$$

Consequently, the joint density function of $\alpha$, $\theta$ and $\beta$ is obtained from Eqs. (21) and (22) is exposed as

$$\pi(\phi) \propto \alpha^{a-1}\theta^{c-1}\beta^{-1}\exp(-b\alpha)\exp(-d\theta), \ a,b,c,d > 0. \tag{23}$$

The posterior density of $\phi$ can be obtained from the likelihood function Eq. (10) and prior distribution in Eq. (23) as follows

$$\pi^*(\phi|\underline{t}) = L(\phi|\underline{t})\pi(\phi) / \int_1^\infty \int_0^\infty \int_0^\infty L(\phi|\underline{t})\pi(\phi) d\alpha d\theta d\beta.$$

The problem of obtaining the closed form of posterior distribution need to formed the normalized constant in the mathematical form, which is more complicated. Hence, we consider the proportional form as follows

$$\begin{aligned} h(\alpha,\theta,\beta|\underline{t}) \propto (\alpha\theta)^{m_1+m_2}\beta^{m_2}\alpha^{a-1}\theta^{c-1}\beta^{-1}\exp\Bigg\{ & -b\alpha - d\theta + (\alpha-1)\left\{\sum_{i=1}^{m_1}\log t_{r_1 i}\right. \\ & \left. + \sum_{i=1}^{m_2}\log t_{r_2 i}\right\} - \sum_{i=1}^{m_1} t_{r_1 i}^{\alpha} - \sum_{i=1}^{m_2} t_{r_2 i}^{\alpha} + (\theta-1)\left\{\sum_{i=1}^{m_1}\log U_{r_1 i} + \sum_{i=1}^{m_2}\log U_{r_2 i}\right\} \\ & + (\beta-1)\sum_{i=1}^{m2}\log\left(1-U_{2i}^{\theta}\right) + \sum_{i=1}^{m_1+1}(r_{1i}-r_{1(i-1)}-1)\log\left[U_{r_{1i}}^{\theta} - U_{r_{1(i-1)}}^{\theta}\right] \\ & + \sum_{i=1}^{m_2+1}(r_{2i}-r_{2(i-1)}-1)\log\left[(1-U_{r_{2(i-1)}}^{\theta})^{\beta} - (1-U_{r_{2i}}^{\theta})^{\beta}\right]\Bigg\}. \end{aligned} \tag{24}$$

The Bayes estimators (BE) of $\alpha$, $\beta$ and $\theta$, using squared error loss function(SEL), can be determined by the following equations as follows

$$\hat{\phi}_{SEL} = E(\phi|\underline{t}) = \frac{\int_1^\infty\int_0^\infty\int_0^\infty \phi L(\alpha,\theta,\beta|\underline{t})\pi(\alpha,\theta,\beta)d\alpha d\theta d\beta}{\int_1^\infty\int_0^\infty\int_0^\infty L(\alpha,\theta,\beta|\underline{t})\pi(\alpha,\theta,\beta)d\alpha d\theta d\beta}. \tag{25}$$

From Eq. (25) the BE of the parameters using the posterior density function with GE loss functions is given by

$$\hat{\phi}_{GE} = [E(\phi^{-q}|\underline{t})]^{\frac{-1}{q}} = \left\{\frac{\int_1^\infty\int_0^\infty\int_0^\infty \phi^{-q} L(\alpha,\theta,\beta|\underline{t})\pi(\alpha,\theta,\beta)d\alpha d\theta d\beta}{\int_1^\infty\int_0^\infty\int_0^\infty L(\alpha,\theta,\beta|\underline{t})\pi(\alpha,\theta,\beta)d\alpha d\theta d\beta}\right\}^{\frac{-1}{q}}. \tag{26}$$

Generally speaking, the ratio of the integrals provided by Eqs. (25) and (26), the two integrals does't be achieved in a closed form, then different approximate methods can be used. The important method can be carry out Bayes estimates of the parameters $\phi = \alpha$, $\theta$ and $\beta$ is MCMC method.

- MCMC Approach

The MCMC method has the priority over the maximum likelihood estimate for constructing the probability intervals from the empirical posterior distribution. About the parameters $\phi = (\alpha, \theta, \beta)$, the posterior is summarized as the MCMC samples, in other words, the simulation Markov chain with some burn-in is the straight computation of the posterior distribution. Then, the estimation problem of the interested parameters or any function of them are computed with generated samples. The theoretical form of posterior distribution determines a suitable scheme of MCMC method that will be used in estimation problem. Gibbs and importance sampling techniques are the important ones with the built of MCMC method.

In this section, we considered the important sampling technique which is suggested by [33] to present point the corresponding credible intervals estimators. The Bayes MCMC method has the property that constructing the credible intervals which unavailable in MLE. The joint posterior distribution of $\alpha$, $\beta$ and $\theta$ is presented by Eq. (24) can be formed as

$$h(\alpha,\theta,\beta|\underline{t}) \propto h_1(\alpha|\theta,\beta,\underline{t}) \times h_2(\beta|\alpha,\theta,\underline{t}) \times h_3(\theta|\alpha,\beta,\underline{t}) \times w(\alpha,\theta,\beta|\underline{t}), \tag{27}$$

then, the joint posterior distribution of $\alpha$, $\theta$ and $\beta$ is written as a product of four function described as follows

**1:** $h_1(\alpha|\beta,\theta,\underline{t})$ is a conditional function of $\alpha$ given $\beta$ and $\theta$, presented by

$$h_1(\alpha|\beta,\theta,\underline{t}) \propto \frac{\alpha^{m_1+m_2+a-1} \exp\left\{-\sum\limits_{i=1}^{m_1} t_{r_1 i}^{\alpha} - \sum\limits_{i=1}^{m_2} t_{r_2 i}^{\alpha} - \sum\limits_{i=1}^{m_1} \log U_{r_1 i} - \sum\limits_{i=1}^{m_2} \log U_{r_2 i}\right\}}{\left(d - \sum\limits_{i=1}^{m_1} \log U_{r_1 i} - \sum\limits_{i=1}^{m_2} \log U_{r_2 i}\right)^{m_1+m_2+c} \left(\sum\limits_{i=1}^{m2} \log\left(1-U_{2i}^{\theta}\right)\right)^{m_2}} \times \exp\left\{-b\alpha + \alpha\left(\sum_{i=1}^{m_1} \log t_{r_1 i} + \sum_{i=1}^{m_2} \log t_{r_2 i}\right)\right\}. \tag{28}$$

**2:** $h_2(\beta|\alpha,\theta,\underline{t})$ is a conditional gamma density function with scale and shape parameters, respectively given by $(m_2)$ and $\left(-\sum\limits_{i=1}^{m2} \log\left(1-U_{2i}^{\theta}\right)\right)$.

**3:** $h_3(\theta|\alpha,\beta,\underline{t})$ is a conditional gamma density function, where the shape and scale parameters given respectively by $(m_1+m_2+c)$ and

$$\left(d - \left(\sum_{i=1}^{m_1} \log U_{r_1 i} + \sum_{i=1}^{m_2} \log U_{r_2 i}\right)\right).$$

**4:** $w(\alpha,\theta,\beta|\underline{t})$ is a weight function given by

$$w(\alpha,\theta,\beta|\underline{t}) = \prod_{i=1}^{m_1+1} \left[U_{r_1 i}^{\theta} - U_{r_1(i-1)}^{\theta}\right]^{(r_{1i}-r_{1(i-1)}-1)} \times \prod_{i=1}^{m_2+1} \left[\left(1-U_{2(i-1)}^{\theta}\right)^{\beta-1} - \left(1-U_{2i}^{\theta}\right)^{\beta-1}\right]^{(r_{2i}-r_{2(i-1)}-1)}. \tag{29}$$

The graph of Eq. (28) is similar to normal distribution. Consequently, we utilized the Metropolis-Hastings manner see [32] together with normal proposal distribution.

- The importance sampling algorithm can be described as follows

Step1: Put the indicator $\kappa = 1$ and start with initial vector $(\alpha^{(0)}, \beta^{(0)}, \theta^{(0)}) = (\hat{\alpha}, \hat{\beta}, \hat{\theta})$.

Step2: Generate $\beta^{(\kappa)}$ from Gamma$\left(m_2, -\sum_{i=1}^{m2} \log\left(1-U_{2i}^{\theta}\right)\right)$.

Step3: Generate $\theta^{(\kappa)}$ from Gamma$\left(m_1+m_2+c, d-\left(\sum_{i=1}^{m_1} \log U_{r_1 i} + \sum_{i=1}^{m_2} \log U_{r_2 i}\right)\right)$.

Step4: Utilising Metropolis-Hastings, generate $\alpha^{(\kappa)}$ using (30) with proposal normal distribution with mean $\alpha^{(\kappa-1)}$ and standard deviation $\sqrt{C_{33}}$ obtained from variances-covariances matrix.

Step5: After obtaining the vector $(\alpha^{(\kappa)}, \beta^{(\kappa)}, \theta^{(\kappa)})$,Compute $w^{(\kappa)}(\alpha^{(\kappa)}, \beta^{(\kappa)}, \theta^{(\kappa)})$ set $\kappa = \kappa + 1$.

Step6: Steps 2-5 is repeated $N$ times.

Step7: The point Bayes estimates under MCMC method of any function $\phi\left(\alpha,\theta,\beta\right)$ is given by

$$E(\phi|\text{data}) = \frac{\sum_{i=M+1}^{N} \phi^{(i)}\left(\alpha^{(i)},\beta^{(i)},\theta^{(i)}\right) w^{(i)}\left(\alpha^{(i)},\beta^{(i)},\theta^{(i)}|\underline{t}\right)}{\sum_{i=M+1}^{N} w^{(i)}\left(\alpha^{(i)},\beta^{(i)},\theta^{(i)}|\underline{t}\right)}, \tag{30}$$

where $\phi\left(\alpha,\ \beta,\ \theta\right) = \alpha,\ \beta,\ \theta$, or any function of the parameters and $M$ is the required number of iteration to reach the stationary distribution. Also the posterior variance of $\phi\left(\alpha,\beta,\theta\right)$ can be written as:

$$\widehat{V}(\phi|\text{data}) = \frac{\sum_{i=M+1}^{N} \left(\phi^{(i)}\left(\alpha^{(i)},\theta^{(i)},\beta^{(i)}\right) - E(\phi|\text{data})\right)^2 w^{(i)}\left(\alpha^{(i)},\beta^{(i)},\theta^{(i)}|\underline{t}\right)}{\sum_{i=M+1}^{N} w^{(i)}\left(\alpha^{(i)},\beta^{(i)},\theta^{(i)}|\underline{t}\right)}. \tag{31}$$

### 4.1 Credible Intervals CIs

The credible intervals of any function $\phi\left(\alpha,\theta,\beta\right)$ is obtained under applied the idea of [33]. The following procedures are performed to get credible CIs of $\alpha$ ,$\theta$ and $\beta$.

**Step 1:** Under the importance sampling technique the values $(\alpha^{(M+1)}, \beta^{(M+1)}, \theta^{(M+1)})$, $(\alpha^{(M+2)}, \beta^{(M+2)}, \theta^{(M+2)})$, ..., $(\alpha^{(N)}, \beta^{(N)}, \theta^{(N)})$ is obtained, then calculate $\phi^{(i)}(\alpha^{(i)}, \beta^{(i)}, \theta^{(i)})$, $i = M+1, M+2, ..., N$.

**Step 2:** The order values of $\phi^{(i)}\left(\alpha^{(i)},\ \beta^{(i)},\ \theta^{(i)}\right)$, $i = M+1,\ M+2,\ ...,\ N$ is given by $\phi_{(i)}\left(\alpha^{(i)},\ \beta^{(i)},\ \theta^{(i)}\right)$, $i = 1,\ 2,\ ...,\ N-M$.

**Step 3:** For $i = 1,\ 2,\ ...,\ N-M$ the ordering values $\phi^{(i)}$ corresponding to the ordering weight function $w_{(i)}$, where

$$w_{(i)} = \frac{\pi^{(i)}\left(\alpha_{(i)}, \theta_{(i)}, \beta_{(i)}|\underline{t}\right)}{\sum\limits_{j=M+1}^{N} \pi^{(i)}\left(\alpha^{(i)}, \theta^{(i)}, \beta^{(i)}|\underline{t}\right)}. \tag{32}$$

**Step 4:** The simulation consistent estimator of cumulative posterior distribution $\hat{\Phi}(\phi|\underline{t})$,

$$\hat{\Phi}(\phi^*|\underline{t}) = \begin{cases} 0, \text{ if } \phi^* < \phi_{(1)} \\ \sum\limits_{j=1}^{i} w_{(i)}, \text{ if } \phi_{(i)} < \phi^* < \phi_{(i+1)} \\ \quad 1, \text{ if } \phi^* > \phi_{(S-M)} \end{cases}. \tag{33}$$

**Step 5:** Let $\phi^{[2\omega]}$ be the $2\omega$-th quantile of $\phi$, *i.e.*

$$\phi^{[2\omega]} = \inf\{\phi;\ \hat{\Phi}(\phi^*|\underline{t}) > 2\omega\} \tag{34}$$

**Step 6:** Then the (1-2$\omega$%) symmetric credible interval for $\phi$, can be constructed.

## 5. APPLICATION

The aim of this section is to assesses the achievement of the estimators and provides the numerical results.

### 5.1 Simulated Data Analysis

The developing procedures discussed in this paper are illustrated through a numerical example described in this section as follows. In generation a simulate data sets, we consider size with the total tested units $n = 50$, groups sizes under normal and accelerate condition $n_1 = n_2 = 25$ and the observed samples $m_1 = m_2 = 15$ are generated from EW distribution with parameters $(\alpha,\ \theta,\ \beta) = (1.5,\ 3,\ 1.6)$. The prior parameters are used under the property $E(\alpha) \simeq \frac{a}{b}$ and $E(\theta) \simeq \frac{c}{d}$, then we used the sets of prior parameters as $\{(a,b),(c,d)\} = \{(3,2),(5,2)\}$The data are summarized in Table 1 under normal conditions (N-C) and accelerated condition (A-C). The Newton Raphson iteration method are employed to compute the MLEs for the point estimates of the parameters as well as 95% approximate confidence intervals (ACIs) are presented in Table 2. Also, in Baysian approach with MCMC method, the chain iteration is employed with 11, 000 times at 1000 values is discarded as 'burn-in'. The results from the Bayes MCMC approach for the point estimates as well as 95% credible intervals (CIs) for the model parameters are constructed in Table 2. The convergence in MCMC method are described with Figs. 1-6 to show the simulation number generated by MCMC method together with the corresponding histogram.

**Table 1. Simulate data set from EW distribution with (1.5, 3.0).**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.7921 | 0.9212 | 0.9244 | 0.9865 | — | — | 1.2078 |
| N-C | 1.4779 | — | — | 1.5023 | — | 1.6263 | 1.6347 |
| | — | 1.7016 | 1.7696 | — | 1.8206 | 1.8685 | — |
| | — | — | 2.5035 | 2.5889 | | | |
| | 0.7921 | — | 0.9244 | 0.9865 | 1.1552 | — | — |
| A-C | 1.4779 | 1.3705 | 1.404 | 1.5023 | 1.5325 | 1.6263 | 1.6347 |
| | — | — | 1.7696 | 1.801 | — | — | 2.1461 |
| | — | 2.3867 | — | — | | | |

**Table 2. MLEs, MSEs, RABs, (95% ACIs) and intervales length.**

| Pa.s | $(.)_{ML}$ | $(.)_{MCMC}$ | 95% ACs | Length | 95% CIs | Length |
|---|---|---|---|---|---|---|
| $\alpha$ | 1.4284 | 1.7198 | (1.17036, 1.68646) | 0.5161 | (1.6327, 2.3116) | 0.6789 |
| $\theta$ | 5.9827 | 4.3336 | (3.38702, 6.57839) | 3.1914 | (2.8423, 5.6626) | 2.8203 |
| $\beta$ | 2.7528 | 1.4932 | (1.44947, 4.05622) | 2.6068 | (0.7132, 5.8148) | 5.1016 |

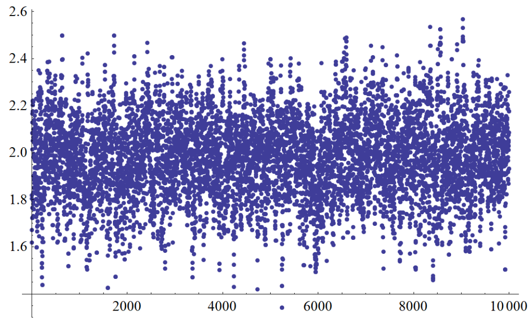


Fig. 1. Simulation number of $\alpha$ generated by MCMC method.

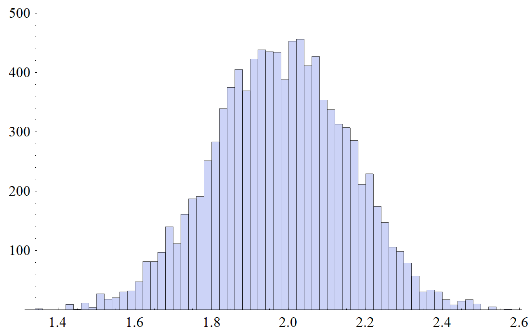


Fig. 2. Histogram of posterior distribution of $\alpha$ generated by MCMC method.

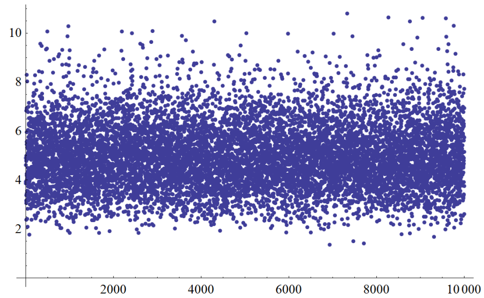


Fig. 3. Simulation number of $\theta$ generated by MCMC method.

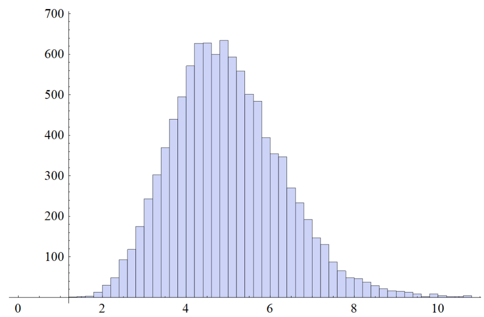


Fig. 4. Histogram of posterior distribution of $\theta$ generated by MCMC method.

### 5.2 Simulation Study

The estimates of the parameters $\alpha$, $\theta$ and $\beta$ of the EW distribution were obtained using multiple Type-II censored scheme under partially constant-stress ALTs. In order to discuss the implementation of different estimators discussed in Sections 2-4, a numerical study is achieved.

The mean square errors (MSEs) and relative absolute biases (RABs) has been used to compared numerically the achievement of the resulting estimators of $(\alpha, \theta, \beta)$. Where, MSE; is ($i.e.$ $MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$) and RABias; is ($i.e.$ $RABias(\hat{\theta}) = \frac{\hat{\theta}-\theta}{\theta}$). The limits of confidence interval and coverage percentage of the unknown parameters were structured at $\gamma = 0.95$. All the computations were conducted in Mathematica (0.9) software. Simulated procedures can be described as follows:

Step 1: With the sample proportion in normal condition $\pi$ the total sample is divided to two sub-grouped with sizes $n_1 = n\pi$ and $n2 = n(1-\pi)$ where $n = n_1 + n_2$.

Step 2: The two samples $t_{j,1} < \ldots < t_{j,n_j}, j = 1, 2$ are generated from normal and accelerated EW distributions Eqs. (2) and (8).

Step 3: 10,000 random samples of sizes $n = 60, 100, 150, 300$ and 500 are generated from $EWD(\alpha, \theta)$ based on multiple Type-II censored sample. The main concept to select different combinations of $n$ is to examine how the MLE and BEs perform for them.

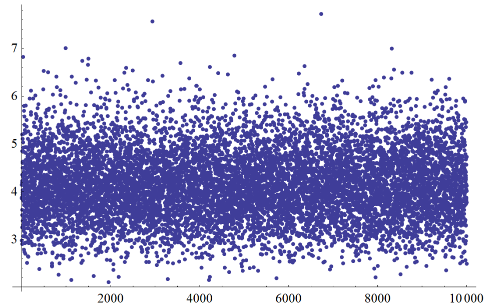


Fig. 5. Simulation number of $\beta$ generated by MCMC method.

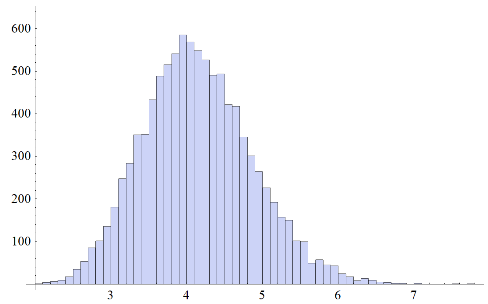


Fig. 6. Histogram of posterior distribution of $\beta$ generated by MCMC method.

Step 4: The population parameters used in the simulation study are case (1) $(\alpha = 0.1, \theta = 2, \beta = 2)$, case (2) $(\alpha = 1.5, \theta = 4, \beta = 2)$ and case (3) $(\alpha = 0.4, \theta = 0.75, \beta = 1.6)$. The selected hyper-parameters are assigned values as $a = 1.0; b = 10; c = 4.0; d = 2.0$.

Step 5: At each sample, we compute the point and interval MLE estimate.

Step 6: In all case, the Bayes estimates with respect to the loss function *SEL* and *GE* of the unknown parameters are constructed for $a = -1, 3, 1$. Simulation results are reported in Tables 1-3 include the $\Upsilon = 0.95$ normal approximation CIs and credible CIs, their lengths of the model parameters and the MSE and RAB for different cases of parameters and sample sizes of $(n, n_1, m_1)$ and $(n, n_2, m_2)$.

Step 7: The MSE and RAB of the estimators for $(\alpha,\theta,\beta)$ for the sample sizes and the three sets of parameters are classified.

### 5.3 Numerical Discussions

Finally, some observations about the performance of the estimated parameters according to Tables 3-5. The following points can be observed

1. We observed that in most of the cases the length of approximate and credible CIs are decreasing when the sample size is increasing.
2. For case (1), the MCMC CIs of $\alpha$, $\theta$ and $\beta$ yield a good outcome than ACIs for interval length.

**Table 3. MSEs, RABs, (95 ACIs and CTs) and length of the MLEs and BEs at ($\alpha$ = 0.1, $\theta$ = 2, $\beta$ = 2).**

| n $(n_1,m_1)$ $(n_2,m_2)$ | Par | ML | SEL | GE c=-1 | GE c=3 | GE c=1 | Approximate CIs L | Credible CIs L |
|---|---|---|---|---|---|---|---|---|
| 60 | $\alpha$ | 0.00030 | 0.00303 | 0.00990 | 0.00894 | 0.00094 | (0.06501, 0.29493) | (0.03087, 0.15111) |
| (30,25) | | 0.20022 | 0.147274 | 0.59143 | 0.59792 | 0.630625 | 0.22990 | 0.13024 |
| (30,25) | $\theta$ | 0.52068 | 0.14100 | 0.17250 | 0.04817 | 0.10122 | (1.69496, 3.2446) | (1.56085, 3.04327) |
| | | 0.23489 | 0.18978 | 0.20767 | 0.10974 | 0.15907 | 1.54963 | 1.48242 |
| | $\beta$ | 2.11489 | 0.03400 | 0.03094 | 0.02123 | 0.02589 | (1.65796, 3.25058) | (1.91153, 3.27968) |
| | | 0.72713 | 0.09418 | 0.04278 | 0.04230 | 0.04254 | 1.5926 | 1.36815 |
| 60 | $\alpha$ | 0.00136 | 0.000009 | 0.00096 | 0.00080 | 0.00088 | (0.04993, 0.27631) | (0.03533, 0.15643) |
| (30,20) | | 0.36879 | 0.02221 | 0.31009 | 0.28294 | 0.29670 | 0.22638 | 0.12110 |
| (30,20) | $\theta$ | 0.70865 | 0.36014 | 0.00369 | 0.02432 | 0.00213 | (1.89192, 3.79171) | (1.25231, 3.05360) |
| | | 0.42091 | 0.30006 | 0.03037 | 0.07797 | 0.02306 | 1.89979 | 1.80129 |
| | $\beta$ | 1.18269 | 0.12627 | 0.00205 | 0.01921 | 0.00841 | (2.75801, 4.32919) | (1.40210, 2.59873) |
| | | 0.77180 | 0.17768 | 0.02265 | 0.06930 | 0.04585 | 1.57118 | 1.19663 |
| 60 | $\alpha$ | 0.00169 | 0.00156 | 0.00012 | 0.00013 | 0.00012 | (0.04816, 0.27973) | (0.02166, 0.16260) |
| (30,15) | | 0.04122 | 0.03785 | 0.00881 | 0.00897 | 0.00885 | 0.23157 | 0.14094 |
| (30,15) | $\theta$ | 1.28526 | 0.87522 | 0.76315 | 0.76117 | 0.45693 | (2.28753, 5.38247) | (0.73921, 2.55203) |
| | | 1.13369 | 1.79548 | 0.92827 | 0.82736 | 0.94105 | 3.09494 | 1.82282 |
| | $\beta$ | 1.23573 | 0.83817 | 0.18467 | 0.20670 | 0.19014 | (1.86171, 3.14963) | (1.01479, 2.41171) |
| | | 1.79881 | 0.82629 | 0.34161 | 0.35597 | 0.34380 | 1.28792 | 1.39692 |
| 100 | $\alpha$ | 0.00004 | 0.00503 | 0.00290 | 0.00272 | 0.00281 | (0.07929, 0.21822) | (0.05231, 0.15602) |
| (50,40) | | 0.06246 | 0.25061 | 0.53807 | 0.52180 | 0.52995 | 0.18893 | 0.10371 |
| (50,40) | $\theta$ | 0.40678 | 0.11643 | 0.00003 | 0.01191 | 0.00325 | (2.01137, 3.26423) | (1.41883, 2.66527) |
| | | 0.31890 | 0.17061 | 0.00257 | 0.05457 | 0.02849 | 1.25285 | 1.24644 |
| | $\beta$ | 1.22756 | 0.02185 | 0.00215 | 0.00895 | 0.00496 | (2.55643, 3.65948) | (1.54768, 2.39294) |
| | | 0.55398 | 0.07391 | 0.02319 | 0.04731 | 0.03519 | 1.10304 | 0.84526 |
| 100 | $\alpha$ | 0.00082 | 0.00009 | 0.00038 | 0.00032 | 0.00035 | (0.05974, 0.28282) | (0.05199, 0.15927) |
| (50,30) | | 0.28718 | 0.09834 | 0.19591 | 0.180223 | 0.188098 | 0.22308 | 0.10728 |
| (50,30) | $\theta$ | 3.41817 | 0.49942 | 0.00956 | 0.05239 | 0.02656 | (2.73569, 4.96197) | (1.28271, 2.65894) |
| | | 0.924414 | 0.353346 | 0.0488887 | 0.11445 | 0.08149 | 2.22628 | 1.37623 |
| | $\beta$ | 2.07754 | 0.11192 | 0.00038 | 0.00695 | 0.00264 | (2.83843, 4.04430) | (1.52023, 2.49862) |
| | | 0.72068 | 0.16728 | 0.00978 | 0.04168 | 0.02568 | 1.20587 | 0.97839 |
| 100 | $\alpha$ | 0.00060 | 0.00012 | 0.00028 | 0.00023 | 0.00026 | (0.06318, 0.28783) | (0.06828, 0.16434) |
| (50,25) | | 0.24497 | 0.11150 | 0.16842 | 0.15292 | 0.16077 | 0.22465 | 0.10606 |
| (50,25) | $\theta$ | 2.73392 | 0.49106 | 0.04867 | 0.00181 | 0.01735 | (2.75199, 5.11269) | (1.44683, 3.18537) |
| | | 0.96617 | 0.35038 | 0.11031 | 0.02129 | 0.06586 | 2.36070 | 1.73854 |
| | $\beta$ | 1.8118 | 0.46728 | 0.10838 | 0.05781 | 0.08113 | (3.03797, 4.3157) | (1.74146, 2.99915) |
| | | 0.83842 | 0.34179 | 0.16461 | 0.12022 | 0.14241 | 1.27774 | 1.25769 |
| 150 | $\alpha$ | 0.00072 | 0.000034 | 0.00053 | 0.00049 | 0.000511 | (0.06366, 0.19259) | (0.07730, 0.15830) |
| (75,60) | | 0.26872 | 0.05819 | 0.23107 | 0.22080 | 0.22598 | 0.12893 | 0.08100 |
| (75,60) | $\theta$ | 0.50789 | 0.40209 | 0.06326 | 0.0977 | 0.07953 | (2.18772, 3.23762) | (1.32379, 2.22705) |
| | | 0.35634 | 0.31706 | 0.12576 | 0.15636 | 0.14100 | 1.04990 | 0.90326 |
| | $\beta$ | 0.81765 | 0.00003 | 0.04384 | 0.05760 | 0.05048 | (2.47753, 3.33095) | (1.48302, 2.13260) |
| | | 0.45212 | 0.00278 | 0.10469 | 0.12000 | 0.11234 | 0.85341 | 0.64958 |
| 150 | $\alpha$ | 0.00083 | 0.00083 | 0.00262 | 0.00252 | 0.00257 | (0.06192, 0.20558) | (0.08348, 0.16880) |
| (75,50) | | 0.28768 | 0.28861 | 0.51217 | 0.50238 | 0.50730 | 0.14366 | 0.08532 |
| (75,50) | $\theta$ | 1.1996 | 0.19563 | 0.40863 | 0.28405 | 0.34378 | (2.46084, 3.72968) | (1.95187, 3.42157) |
| | | 0.54763 | 0.22115 | 0.31962 | 0.26648 | 0.29316 | 1.26884 | 1.46970 |
| | $\beta$ | 1.58261 | 0.17328 | 0.02298 | 0.01169 | 0.01686 | (2.78489, 3.73115) | (1.75642, 2.59183) |
| | | 0.62901 | 0.20814 | 0.07579 | 0.05406 | 0.06493 | 0.94626 | 0.83541 |
| 150 | $\alpha$ | 0.00149 | 0.00004 | 0.00040 | 0.00035 | 0.00038 | (0.03545, 0.22075) | (0.10268, 0.16772) |
| (75,38) | | 0.385527 | 0.06532 | 0.20116 | 0.18770 | 0.19447 | 0.1853 | 0.06504 |
| (75,38) | $\theta$ | 4.8092 | 0.44042 | 0.02174 | 0.06041 | 0.03854 | (1.22526, 3.96071) | (1.31097, 2.49221) |
| | | 1.09649 | 0.331821 | 0.07373 | 0.12289 | 0.09816 | 1.43545 | 1.18124 |
| | $\beta$ | 1.62009 | 0.01243 | 0.02718 | 0.04592 | 0.03591 | (1.80679, 3.73885) | (1.44167, 2.28254) |
| | | 1.46003 | 0.05576 | 0.08244 | 0.10714 | 0.09475 | 1.93206 | 0.84137 |
| 300 | $\alpha$ | 0.00108 | 0.00002 | 0.00012 | 0.00012 | 0.00011 | (0.061157, 0.18291) | (0.09177, 0.14977) |
| (150,125) | | 0.32936 | 0.04308 | 0.10741 | 0.10348 | 0.10545 | 0.121753 | 0.05800 |
| (150,125) | $\theta$ | 0.528751 | 0.14624 | 0.01464 | 0.02317 | 0.01866 | (2.07409, 2.78021) | (1.5569, 2.23128) |
| | | 0.363576 | 0.191206 | 0.0605049 | 0.07611 | 0.06829 | 0.70612 | 0.67438 |
| | $\beta$ | 0.76847 | 0.00007 | 0.01654 | 0.02084 | 0.01863 | (2.17896, 2.88926) | (1.64324, 2.11493) |
| | | 0.39205 | 0.00423 | 0.06429 | 0.07219 | 0.06824 | 0.7103 | 0.47169 |
| 300 | $\alpha$ | 0.00159 | 0.00004 | 0.00029 | 0.00028 | 0.00029 | (0.05436, 0.186582) | (0.09065, 0.15247) |
| (150,100) | | 0.39912 | 0.06660 | 0.17264 | 0.16736 | 0.16999 | 0.132222 | 0.06182 |
| (150,100) | $\theta$ | 2.42815 | 0.29469 | 0.03942 | 0.05588 | 0.04729 | (2.02079, 3.09571) | (1.465, 2.18448) |
| | | 0.77913 | 0.27143 | 0.09927 | 0.11820 | 0.10873 | 1.07492 | 0.71948 |
| | $\beta$ | 1.6362 | 0.00065 | 0.01464 | 0.01976 | 0.01710 | (2.04813, 2.81015) | (1.62203, 2.15317) |
| | | 0.63957 | 0.01274 | 0.06049 | 0.07029 | 0.065392 | 0.76202 | 0.53114 |
| 300 | $\alpha$ | 0.00085 | 0.00035 | 0.00055 | 0.00048 | 0.00050 | (0.06331, 0.19854) | (0.09811, 0.15791) |
| (150,75) | | 0.29072 | 0.18649 | 0.22924 | 0.21959 | 0.22440 | 0.13523 | 0.05980 |
| (150,75) | $\theta$ | 1.53167 | 0.82928 | 0.37972 | 0.43103 | 0.40487 | (1.91809, 3.57069) | (1.07714, 2.13450) |
| | | 0.77219 | 0.45532 | 0.30811 | 0.32826 | 0.31815 | 1.6526 | 1.05736 |
| | $\beta$ | 0.69406 | 0.15667 | 0.21874 | 0.24066 | 0.22956 | (1.93048, 3.13573) | (1.28423, 2.19889) |
| | | 0.41655 | 0.197909 | 0.23385 | 0.24528 | 0.23956 | 1.20525 | 0.91466 |
| 500 | $\alpha$ | 0.00071 | 0.00037 | 0.00067 | 0.00066 | 0.00066 | (0.06843, 0.17825) | (0.09811, 0.13791) |
| (250,210) | | 0.26662 | 0.19201 | 0.25841 | 0.25614 | 0.25728 | 0.10982 | 0.03980 |
| (250,210) | $\theta$ | 0.46630 | 0.01124 | 0.01271 | 0.00844 | 0.01047 | (2.03099, 2.63474) | (1.83268, 2.40953) |
| | | 0.34142 | 0.05300 | 0.05636 | 0.04595 | 0.05116 | 0.60375 | 0.57685 |
| | $\beta$ | 0.31505 | 0.00005 | 0.00666 | 0.00513 | 0.00586 | (2.10155, 2.59236) | (1.88853, 2.28507) |
| | | 0.47348 | 0.00302 | 0.04079 | 0.03579 | 0.03829 | 0.49081 | 0.39654 |
| 500 | $\alpha$ | 0.00129 | 0.00009 | 0.00039 | 0.00038 | 0.00039 | (0.0552, 0.1740) | (0.09058, 0.1417) |
| (250,170) | | 0.35945 | 0.09915 | 0.19795 | 0.19497 | 0.19646 | 0.1188 | 0.05112 |
| (250,170) | $\theta$ | 1.33608 | 0.14053 | 0.00679 | 0.01116 | 0.00884 | (2.1285, 2.9339) | (1.6126, 2.1883) |
| | | 0.76421 | 0.18744 | 0.04119 | 0.05283 | 0.04701 | 0.8054 | 0.5757 |
| | $\beta$ | 1.15722 | 0.00095 | 0.01245 | 0.01517 | 0.01378 | (2.0591, 2.6503) | (1.6162, 2.0110) |
| | | 0.53787 | 0.01540 | 0.05578 | 0.06159 | 0.05869 | 0.5912 | 0.3948 |
| 500 | $\alpha$ | 0.00134 | 0.00039 | 0.00083 | 0.00080 | 0.00082 | (0.05839, 0.18836) | (0.10795, 0.16943) |
| (250,125) | | 0.36626 | 0.19953 | 0.28815 | 0.28346 | 0.28581 | 0.12997 | 0.06148 |
| (250,125) | $\theta$ | 2.67418 | 0.21207 | 0.02579 | 0.02667 | 0.02601 | (2.00284, 3.06405) | (1.48874, 2.12641) |
| | | 0.92917 | 0.34982 | 0.10139 | 0.11646 | 0.10890 | 1.06121 | 0.63767 |
| | $\beta$ | 1.35205 | 0.02694 | 0.06667 | 0.07445 | 0.07051 | (2.01415, 3.0114) | (1.52587, 2.17005) |
| | | 0.58139 | 0.08206 | 0.12910 | 0.13643 | 0.13277 | 1.09725 | 0.44418 |

**Table 4. MSEs, RABs, (95 ACIs and CTs) and length of the MLEs and BEs at ($\alpha = 1.5, \theta = 4, \beta = 2$).**

| n $(n_1,m_1)$ $(n_2,m_2)$ | Par | ML | SEL | GE c=-1 | GE c=3 | GE c=1 | Approximate CIs L | Credible CIs L |
|---|---|---|---|---|---|---|---|---|
| 60 | $\alpha$ | 0.00162 | 0.02995 | 0.50240 | 0.47158 | 0.48695 | (1.12482, 1.79475) | (1.30536, 2.20901) |
| (30,25) | | 0.02681 | 0.11538 | 0.47253 | 0.45781 | 0.46521 | 0.66993 | 0.90365 |
| (30,25) | $\theta$ | 1.10077 | 0.03147 | 0.01306 | 0.00158 | 0.00140 | (3.89874, 6.19961) | (3.08633, 5.29473) |
| | | 0.26229 | 0.04435 | 0.02857 | 0.00992 | 0.00936 | 2.30087 | 2.20840 |
| | $\beta$ | 0.22442 | 0.00006 | 4.99816 | 3.56074 | 4.25461 | (1.71678, 3.23068) | (1.72901, 5.05202) |
| | | 0.23686 | 0.00372 | 1.11783 | 0.94350 | 1.03134 | 1.5139 | 3.32301 |
| 60 | $\alpha$ | 0.11606 | 0.104794 | 0.12916 | 0.11430 | 0.12170 | (0.95362, 1.66504) | (1.39530, 2.32876) |
| (30,20) | | 0.22712 | 0.21581 | 0.23959 | 0.22539 | 0.23257 | 0.71142 | 0.93346 |
| (30,20) | $\theta$ | 1.20538 | 0.09785 | 0.01467 | 0.00541 | 0.00056 | (3.95818, 6.23762) | (2.99205, 5.47685) |
| | | 0.27447 | 0.07820 | 0.03028 | 0.01838 | 0.00594 | 2.27944 | 2.4848 |
| | $\beta$ | 1.12180 | 0.04007 | 5.2223 | 3.42740 | 4.28893 | (2.02041, 4.09787) | (1.61813, 5.32175) |
| | | 0.52957 | 0.10009 | 1.14262 | 0.92566 | 1.03549 | 2.07746 | 3.70362 |
| 60 | $\alpha$ | 0.38637 | 0.15037 | 0.37926 | 0.33391 | 0.35656 | (0.68308, 1.77375) | (1.32606, 2.39895) |
| (30,15) | | 0.41439 | 0.34963 | 0.41056 | 0.38523 | 0.39808 | 1.09067 | 1.07289 |
| (30,15) | $\theta$ | 1.31752 | 0.16702 | 0.10469 | 0.28935 | 0.18524 | (3.58456, 5.54241) | (2.56154, 4.99019) |
| | | 0.34087 | 0.10217 | 0.08089 | 0.13448 | 0.10760 | 1.95786 | 2.42865 |
| | $\beta$ | 1.3976 | 1.36247 | 17.2062 | 10.86730 | 13.8974 | (2.01934, 4.07595) | (1.43215, 5.61770) |
| | | 0.67238 | 0.58363 | 2.07402 | 1.64828 | 1.86396 | 2.05662 | 4.18555 |
| 100 | $\alpha$ | 0.14562 | 0.00622 | 0.07375 | 0.06702 | 0.07036 | (0.96435, 1.57245) | (1.36501, 1.98068) |
| (50,40) | | 0.25440 | 0.05256 | 0.18105 | 0.17259 | 0.17684 | 0.60809 | 0.61567 |
| (50,40) | $\theta$ | 0.01947 | 0.59656 | 0.51971 | 0.63587 | 0.57637 | (3.42125, 4.85782) | (2.61841, 4.21996) |
| | | 0.03488 | 0.19309 | 0.18023 | 0.19935 | 0.18980 | 1.43657 | 1.60155 |
| | $\beta$ | 0.25797 | 0.00005 | 0.95718 | 0.68694 | 0.81749 | (2.12428, 3.39854) | (2.11893, 3.94944) |
| | | 0.18070 | 0.00756 | 0.48918 | 0.41440 | 0.45208 | 1.27426 | 1.83051 |
| 100 | $\alpha$ | 0.20182 | 0.13167 | 0.56533 | 0.53885 | 0.55211 | (0.89434, 1.50717) | (1.76664, 2.52908) |
| (50,33) | | 0.29950 | 0.24191 | 0.50126 | 0.48938 | 0.49536 | 0.61283 | 0.76244 |
| (50,33) | $\theta$ | 0.31038 | 0.13437 | 0.25362 | 0.36612 | 0.30709 | (3.79753, 5.31671) | (2.71910, 4.36543) |
| | | 0.13928 | 0.09164 | 0.12590 | 0.15127 | 0.13854 | 1.51918 | 1.64633 |
| | $\beta$ | 1.33103 | 0.00743 | 3.67975 | 2.8285 | 3.24119 | (2.38294, 3.92447) | (2.71746, 5.38836) |
| | | 0.57685 | 0.07849 | 0.95913 | 0.84091 | 0.90017 | 1.54154 | 2.6709 |
| 100 | $\alpha$ | 0.22845 | 0.18499 | 0.18245 | 0.16810 | 0.17524 | (0.87222, 1.57186) | (1.47935, 2.28137) |
| (50,25) | | 0.31864 | 0.28141 | 0.28476 | 0.27334 | 0.27908 | 0.69964 | 0.80202 |
| (50,25) | $\theta$ | 1.64209 | 0.03825 | 0.02828 | 0.00005 | 0.00774 | (4.38787, 6.17501) | (3.11808, 5.38588) |
| | | 0.32036 | 0.04889 | 0.04205 | 0.00171 | 0.02199 | 1.78714 | 2.2678 |
| | $\beta$ | 1.16760 | 0.02803 | 4.78373 | 3.42330 | 4.07872 | (2.86527, 5.21767) | (2.69897, 5.97161) |
| | | 0.62073 | 0.14481 | 1.09359 | 0.92511 | 1.00979 | 2.35241 | 2.68691 |
| 150 | $\alpha$ | 0.01259 | 0.30934 | 0.55083 | 0.54047 | 0.54565 | (1.34400, 1.73159) | (1.46857, 2.12307) |
| (75,60) | | 0.074800 | 0.37079 | 0.49479 | 0.49011 | 0.49245 | 0.38759 | 0.65450 |
| (75,60) | $\theta$ | 1.33915 | 0.49399 | 0.22389 | 0.159460 | 0.19036 | (3.42719, 4.88724) | (3.71126, 5.29758) |
| | | 0.28930 | 0.17571 | 0.11829 | 0.09983 | 0.10908 | 1.46005 | 1.58632 |
| | $\beta$ | 0.44688 | 0.00004 | 6.54248 | 5.78297 | 6.15807 | (2.17388, 3.16310) | (3.47397, 5.77675) |
| | | 0.33425 | 0.00556 | 1.27891 | 1.20239 | 1.24077 | 0.98922 | 2.30278 |
| 150 | $\alpha$ | 0.12488 | 0.11061 | 0.25235 | 0.24416 | 0.24824 | (1.01953, 1.67369) | (1.32162, 2.18158) |
| (75,50) | | 0.23559 | 0.22173 | 0.33489 | 0.32941 | 0.33216 | 0.65416 | 0.85996 |
| (75,50) | $\theta$ | 1.10022 | 0.79942 | 0.01098 | 0.00051 | 0.00407 | (3.35141, 4.94641) | (3.34247, 4.95593) |
| | | 0.26223 | 0.22353 | 0.02620 | 0.00562 | 0.01594 | 1.59500 | 1.61346 |
| | $\beta$ | 0.94362 | 0.04275 | 6.11806 | 6.04863 | 6.57487 | (2.56715, 3.91770) | (3.59814, 6.22537) |
| | | 0.62121 | 0.17938 | 1.12461 | 1.12746 | 1.17612 | 1.35055 | 2.62723 |
| 150 | $\alpha$ | 0.16571 | 0.06717 | 0.25243 | 0.24217 | 0.2473040 | (0.96809, 1.61777) | (1.2995, 2.19825) |
| (75,38) | | 0.27138 | 0.17278 | 0.33495 | 0.32807 | 0.33153 | 0.64968 | 0.89875 |
| (75,38) | $\theta$ | 2.51638 | 0.07873 | 0.05233 | 0.01418 | 0.03027 | (3.80412, 5.36850) | (3.33111, 5.2144) |
| | | 0.39658 | 0.07015 | 0.05719 | 0.02977 | 0.04350 | 1.56438 | 1.88329 |
| | $\beta$ | 1.19858 | 0.16145 | 7.30326 | 6.67929 | 7.47904 | (3.09327, 5.00482) | (3.53391, 5.82571) |
| | | 0.80245 | 0.63396 | 1.25250 | 1.38558 | 1.45594 | 1.91155 | 2.29180 |
| 300 | $\alpha$ | 0.01499 | 0.31672 | 0.49314 | 0.48876 | 0.49095 | (1.27518, 1.55584) | (1.48956, 2.01867) |
| (150,125) | | 0.15633 | 0.37518 | 0.46816 | 0.46608 | 0.46712 | 0.28065 | 0.52911 |
| (150,125) | $\theta$ | 1.62087 | 0.69476 | 0.48380 | 0.43246 | 0.45777 | (4.75133, 5.79494) | (4.12796, 5.2981) |
| | | 0.31828 | 0.20838 | 0.17389 | 0.16440 | 0.16916 | 1.04362 | 1.17014 |
| | $\beta$ | 0.41069 | 0.00659 | 6.90697 | 6.48301 | 6.69377 | (2.34982, 3.02232) | (4.01907, 5.68644) |
| | | 0.24303 | 0.09269 | 1.10597 | 1.16775 | 1.18688 | 0.67250 | 1.66737 |
| 300 | $\alpha$ | 0.05942 | 0.26073 | 0.60674 | 0.60048 | 0.60361 | (0.97945 , 1.55011) | (1.34483, 2.11148) |
| (150,100) | | 0.15015 | 0.34041 | 0.51929 | 0.51660 | 0.51795 | 0.57066 | 0.76665 |
| (150,100) | $\theta$ | 1.0702 | 0.11509 | 0.09045 | 0.06658 | 0.07807 | (4.54874, 5.52026) | (3.72673, 4.91530) |
| | | 0.25863 | 0.08481 | 0.07519 | 0.06451 | 0.06985 | 0.97152 | 1.18857 |
| | $\beta$ | 1.07489 | 0.71653 | 6.95078 | 6.47583 | 6.71089 | (2.93838, 3.94252) | (3.77037, 5.59057) |
| | | 0.72022 | 0.88268 | 1.31822 | 1.27238 | 1.29527 | 1.00414 | 1.82020 |
| 300 | $\alpha$ | 0.24357 | 0.18467 | 0.24355 | 0.23908 | 0.24131 | (1.02198, 1.79096) | (1.26523, 2.11233) |
| (150,75) | | 0.23319 | 0.28649 | 0.32901 | 0.32597 | 0.32749 | 0.76898 | 0.84710 |
| (150,75) | $\theta$ | 2.05409 | 3.66192 | 1.22137 | 1.07520 | 1.14721 | (4.90699, 5.95944) | (4.32804, 5.93968) |
| | | 0.35830 | 0.47840 | 0.27629 | 0.25923 | 0.26777 | 1.05245 | 1.61164 |
| | $\beta$ | 0.95915 | 0.85210 | 8.10569 | 7.30497 | 7.70158 | (3.49097, 4.79463) | (3.95081, 6.21329) |
| | | 0.97140 | 0.98046 | 1.21087 | 1.24092 | 1.27492 | 1.30366 | 2.26248 |
| 500 | $\alpha$ | 0.00135 | 0.13943 | 0.27519 | 0.27317 | 0.27418 | (1.38654, 1.51388) | (1.43729, 1.81021) |
| (250,210) | | 0.01508 | 0.24893 | 0.34972 | 0.34844 | 0.34908 | 0.12734 | 0.37292 |
| (250,210) | $\theta$ | 1.34949 | 0.18744 | 0.16236 | 0.14555 | 0.15385 | (3.76473, 4.55862) | (3.06014, 4.34316) |
| | | 0.29042 | 0.10824 | 0.10074 | 0.09538 | 0.09806 | 0.793887 | 1.28302 |
| | $\beta$ | 0.00593 | 0.16167 | 4.86821 | 4.69069 | 4.77914 | (2.11556, 2.56131) | (2.65066, 3.79852) |
| | | 0.06326 | 0.10672 | 1.10320 | 1.08290 | 1.09306 | 0.44575 | 1.14786 |
| 500 | $\alpha$ | 0.02584 | 0.18316 | 0.30706 | 0.30475 | 0.30590 | (1.263621, 1.58593) | (1.46348, 2.04528) |
| (250,170) | | 0.31682 | 0.28532 | 0.36942 | 0.36803 | 0.36872 | 0.322309 | 0.58180 |
| (250,170) | $\theta$ | 1.88490 | 1.74140 | 1.07565 | 1.01479 | 1.04493 | (3.96800, 4.77783) | (3.52325, 5.00317) |
| | | 0.34323 | 0.32991 | 0.25928 | 0.25184 | 0.25556 | 0.809829 | 1.47992 |
| | $\beta$ | 0.83982 | 0.93556 | 8.60100 | 8.23506 | 8.41750 | (2.10182, 2.89138) | (2.36115, 3.88824) |
| | | 0.74830 | 0.87303 | 1.04927 | 1.01946 | 1.03440 | 0.78956 | 1.52709 |
| 500 | $\alpha$ | 0.10140 | 0.26592 | 0.421586 | 0.41758 | 0.41959 | (0.93431, 1.56290) | (1.33540, 2.25891) |
| (250,125) | | 0.13426 | 0.34378 | 0.43286 | 0.43080 | 0.43184 | 0.628587 | 0.92351 |
| (250,125) | $\theta$ | 1.60397 | 0.46693 | 0.48107 | 0.43014 | 0.45526 | (3.87265, 5.06031) | (3.12471, 5.29481) |
| | | 0.31662 | 0.170831 | 0.17340 | 0.16396 | 0.16868 | 1.187661 | 2.17010 |
| | $\beta$ | 0.86801 | 0.98601 | 8.84330 | 8.2888 | 8.5641 | (2.96930, 4.01404) | (2.41442, 4.26474) |
| | | 0.83708 | 0.93568 | 1.14646 | 1.1038 | 1.12512 | 1.04474 | 1.85032 |

**Table 5. MSEs, RABs, (95 ACIs and CTs) and length of the MLEs and BEs at ($\alpha = 0.4, \theta = 0.75, \beta = 1.6$).**

| n $(n_1, m_1)$ $(n_2, m_2)$ | Par | ML | SEL | GE c=-1 | GE c=3 | GE c=1 | Approximate CIs L | Credible CIs L |
|---|---|---|---|---|---|---|---|---|
| 60 | $\alpha$ | 0.00531 | 0.02892 | 0.03198 | 0.03691 | 0.03443 | (0.22091, 0.43333) | (0.15320, 0.30418) |
| (30,25) | | 0.18220 | 0.42513 | 0.44708 | 0.48028 | 0.46388 | 0.21242 | 0.15097 |
| (30,25) | $\theta$ | 0.02288 | 0.04796 | 0.08180 | 0.05058 | 0.06527 | (0.54977, 1.25275) | (0.71893, 1.41819) |
| | | 0.20168 | 0.29199 | 0.38135 | 0.29988 | 0.34063 | 0.70299 | 0.69926 |
| | $\beta$ | 0.19452 | 0.26889 | 0.32707 | 0.45702 | 0.38971 | (1.28840, 2.79370) | (0.63377, 1.53793) |
| | | 0.27566 | 0.32409 | 0.35744 | 0.42252 | 0.39017 | 1.50530 | 0.90416 |
| 60 | $\alpha$ | 0.03441 | 0.02161 | 0.03132 | 0.03776 | 0.03453 | (0.14161, 0.38739) | (0.14716, 0.32316) |
| (30,20) | | 0.46374 | 0.36751 | 0.44245 | 0.48579 | 0.46458 | 0.24578 | 0.17599 |
| (30,20) | $\theta$ | 0.46514 | 0.06742 | 0.07035 | 0.03479 | 0.05109 | (0.93202, 1.93200) | (0.66291, 1.44642) |
| | | 0.90935 | 0.34619 | 0.35366 | 0.24872 | 0.30137 | 0.99998 | 0.78351 |
| | $\beta$ | 0.70478 | 0.25434 | 0.31995 | 0.49243 | 0.40165 | (1.54594, 3.33308) | (0.58618, 1.63510 ) |
| | | 0.52469 | 0.31520 | 0.35353 | 0.43858 | 0.39609 | 1.78714 | 1.04892 |
| 60 | $\alpha$ | 0.01639 | 0.01624 | 0.04045 | 0.04739 | 0.04388 | (0.17459, 0.46933) | (0.12932, 0.32848) |
| (30,15) | | 0.32009 | 0.31863 | 0.50278 | 0.54424 | 0.52368 | 0.29474 | 0.18917 |
| (30,15) | $\theta$ | 0.23286 | 0.04033 | 0.19624 | 0.11996 | 0.15567 | (0.76280, 1.80230) | (0.76885, 1.71534) |
| | | 0.64340 | 0.26777 | 0.59065 | 0.46179 | 0.52606 | 1.09395 | 0.94649 |
| | $\beta$ | 2.65350 | 0.45257 | 0.38012 | 0.60172 | 0.48511 | (1.84049, 4.61743) | (0.52004, 1.61744) |
| | | 1.01810 | 0.42046 | 0.38534 | 0.48482 | 0.43531 | 2.77694 | 1.09739 |
| 100 | $\alpha$ | 0.01076 | 0.00947 | 0.01647 | 0.01918 | 0.01781 | (0.22857, 0.41639) | (0.20566, 0.35195) |
| (50,40) | | 0.25932 | 0.24331 | 0.32082 | 0.34622 | 0.33364 | 0.19539 | 0.14629 |
| (50,40) | $\theta$ | 0.25145 | 0.05421 | 0.08831 | 0.06631 | 0.07693 | (0.92465, 1.57825) | (0.78424, 1.34915) |
| | | 0.66859 | 0.31044 | 0.39623 | 0.34335 | 0.36982 | 0.65359 | 0.56491 |
| | $\beta$ | 0.51040 | 0.31550 | 0.31585 | 0.39451 | 0.35431 | (1.73308, 2.89578) | (0.71538 , 1.44663) |
| | | 0.44652 | 0.35106 | 0.35125 | 0.39256 | 0.37202 | 1.16270 | 0.73125 |
| 100 | $\alpha$ | 0.00656 | 0.01402 | 0.02468 | 0.02863 | 0.02663 | (0.22363, 0.42445) | (0.17423, 0.32360) |
| (50,33) | | 0.20239 | 0.29605 | 0.39277 | 0.42301 | 0.40797 | 0.20083 | 0.14937 |
| (50,33) | $\theta$ | 0.08133 | 0.01759 | 0.06333 | 0.04152 | 0.05186 | (0.70496, 1.36539) | (0.722258, 1.32206) |
| | | 0.38024 | 0.17688 | 0.33554 | 0.27169 | 0.30363 | 0.66043 | 0.599802 |
| | $\beta$ | 0.99944 | 0.34333 | 0.32992 | 0.43283 | 0.37950 | (1.88132, 3.31812) | (0.66652, 1.48030) |
| | | 0.62483 | 0.36621 | 0.35899 | 0.41118 | 0.38502 | 1.43680 | 0.81378 |
| 100 | $\alpha$ | 0.03801 | 0.0102591 | 0.036715 | 0.0417506 | 0.0392562 | (0.152512, 0.45759) | (0.146598, 0.293443) |
| (50,25) | | 0.48738 | 0.25322 | 0.47903 | 0.51082 | 0.49533 | 0.20508 | 0.14685 |
| (50,25) | $\theta$ | 0.61380 | 0.01265 | 0.11295 | 0.07291 | 0.09199 | (1.13297, 1.93394) | (0.74674, 1.48137) |
| | | 1.04461 | 0.14997 | 0.44810 | 0.36003 | 0.40441 | 0.80098 | 0.73463 |
| | $\beta$ | 4.25613 | 0.47171 | 0.36683 | 0.50301 | 0.43184 | (2.51484, 4.81123) | (0.60252, 1.49364) |
| | | 1.2894 | 0.429255 | 0.378539 | 0.443266 | 0.410715 | 2.29639 | 0.89112 |
| 150 | $\alpha$ | 0.00628 | 0.00932 | 0.01604 | 0.01806 | 0.01704 | (0.25755, 0.38394) | (0.21460, 0.33977) |
| (75,60) | | 0.19814 | 0.24131 | 0.31662 | 0.33597 | 0.32629 | 0.12639 | 0.12517 |
| (75,60) | $\theta$ | 0.14873 | 0.03222 | 0.04763 | 0.03661 | 0.04191 | (0.88221, 1.38910) | (0.76027, 1.21028) |
| | | 0.51421 | 0.23935 | 0.29100 | 0.25511 | 0.27297 | 0.50688 | 0.45001 |
| | $\beta$ | 0.57359 | 0.32927 | 0.369708 | 0.42482 | 0.39685 | (1.89173, 2.82299) | (0.72603, 1.31367) |
| | | 0.47335 | 0.35864 | 0.38002 | 0.40737 | 0.39373 | 0.93126 | 0.58764 |
| 150 | $\alpha$ | 0.03853 | 0.01007 | 0.02199 | 0.02410 | 0.02305 | (0.16620, 0.34121) | (0.19788, 0.31578) |
| (75,50) | | 0.49073 | 0.25088 | 0.37078 | 0.38813 | 0.37951 | 0.17501 | 0.11790 |
| (75,50) | $\theta$ | 0.89388 | 0.07185 | 0.13906 | 0.11624 | 0.12739 | (1.36736, 2.02355) | (0.87567, 1.39952) |
| | | 1.26061 | 0.35741 | 0.49721 | 0.45458 | 0.47590 | 0.65619 | 0.52385 |
| | $\beta$ | 1.48705 | 0.34109 | 0.26722 | 0.32781 | 0.29676 | (2.19486, 3.44403) | (0.77051, 1.45353) |
| | | 0.76215 | 0.36501 | 0.32308 | 0.35784 | 0.34047 | 1.24917 | 0.68302 |
| 150 | $\alpha$ | 0.039921 | 0.00658 | 0.01504 | 0.01725 | 0.01612 | (0.16207, 0.33833 ) | (0.21374, 0.35134) |
| (75,38) | | 0.49950 | 0.20282 | 0.30659 | 0.32831 | 0.31738 | 0.17626 | 0.13759 |
| (75,38) | $\theta$ | 0.83497 | 0.04058 | 0.14774 | 0.11840 | 0.13267 | (1.33419, 1.99334) | (0.85671, 1.45574) |
| | | 1.21836 | 0.26860 | 0.51249 | 0.45879 | 0.48566 | 0.65915 | 0.59903 |
| | $\beta$ | 3.76714 | 0.17357 | 0.25002 | 0.32498 | 0.28628 | (2.70838, 4.37345) | (0.75478, 1.52029) |
| | | 1.21307 | 0.26039 | 0.31252 | 0.35629 | 0.33441 | 1.66506 | 0.76551 |
| 300 | $\alpha$ | 0.01222 | 0.01734 | 0.01849 | 0.01936 | 0.01893 | (0.24919, 0.32975) | (0.22649, 0.30622) |
| (150,125) | | 0.27631 | 0.32920 | 0.34000 | 0.34783 | 0.34393 | 0.08055 | 0.07973 |
| (150,125) | $\theta$ | 0.12288 | 0.03114 | 0.05358 | 0.04779 | 0.05065 | (0.92797, 1.27311) | (0.83088, 1.14346) |
| | | 0.46739 | 0.23527 | 0.30862 | 0.29148 | 0.30007 | 0.34515 | 0.31258 |
| | $\beta$ | 0.25740 | 0.25676 | 0.30718 | 0.33136 | 0.31916 | (1.83506, 2.37963) | (0.84936, 1.26164) |
| | | 0.317091 | 0.316699 | 0.3464 | 0.359772 | 0.353087 | 0.54458 | 0.41228 |
| 300 | $\alpha$ | 0.02407 | 0.00352 | 0.00688 | 0.00765 | 0.00726 | (0.21307, 0.27666) | (0.26621, 0.37476) |
| (150,100) | | 0.38785 | 0.14829 | 0.20731 | 0.21859 | 0.21296 | 0.06359 | 0.10856 |
| (150,100) | $\theta$ | 0.74119 | 0.04995 | 0.09698 | 0.08691 | 0.09188 | (1.38585, 1.83600) | (0.88712, 1.2533) |
| | | 1.14790 | 0.29798 | 0.41521 | 0.39307 | 0.40415 | 0.45015 | 0.36619 |
| | $\beta$ | 1.98213 | 0.31122 | 0.32278 | 0.35303 | 0.33773 | (2.53424, 3.48152) | (06.81659, 1.27312) |
| | | 0.87992 | 0.34867 | 0.35508 | 0.37135 | 0.36322 | 0.94728 | 0.45653 |
| 300 | $\alpha$ | 0.03126 | 0.00275 | 0.01625 | 0.01761 | 0.01693 | (0.19213, 0.25427) | (0.22416, 0.33064) |
| (150,75) | | 0.44201 | 0.13099 | 0.31873 | 0.33179 | 0.32531 | 0.06214 | 0.10648 |
| (150,75) | $\theta$ | 0.85623 | 0.01887 | 0.06996 | 0.05944 | 0.06461 | (1.43762, 1.91304) | (0.82447, 1.2281) |
| | | 1.23377 | 0.18316 | 0.35267 | 0.32508 | 0.338907 | 0.47542 | 0.40363 |
| | $\beta$ | 5.75215 | 0.46115 | 0.36971 | 0.41245 | 0.39085 | (3.28223, 4.71450) | (0.75379, 1.26424) |
| | | 1.49898 | 0.424427 | 0.380025 | 0.401387 | 0.390736 | 1.43227 | 0.51046 |
| 500 | $\alpha$ | 0.00831 | 0.00632 | 0.00960 | 0.01002 | 0.00981 | (0.27766, 0.34000) | (0.26853, 0.33856) |
| (250,210) | | 0.22792 | 0.19873 | 0.24496 | 0.25022 | 0.24759 | 0.06234 | 0.07003 |
| (250,210) | $\theta$ | 0.27384 | 0.04465 | 0.06385 | 0.06006 | 0.06194 | (1.12718, 1.41942) | (0.88485, 1.12722) |
| | | 0.69773 | 0.28176 | 0.33691 | 0.32676 | 0.33184 | 0.29224 | 0.24237 |
| | $\beta$ | 0.48588 | 0.36409 | 0.35528 | 0.37033 | 0.36275 | (2.0508, 2.5433) | (0.85387, 1.16289) |
| | | 0.43566 | 0.37713 | 0.37254 | 0.38034 | 0.37643 | 0.49249 | 0.30902 |
| 500 | $\alpha$ | 0.02072 | 0.00673 | 0.014288 | 0.01485 | 0.01456 | (0.22968, 0.28244) | (0.24627, 0.31702) |
| (250,170) | | 0.35985 | 0.20506 | 0.29879 | 0.30463 | 0.30172 | 0.05276 | 0.07075 |
| (250,170) | $\theta$ | 0.59969 | 0.05509 | 0.08713 | 0.08166 | 0.08437 | (1.35459, 1.69420) | (0.91254, 1.18894) |
| | | 1.03252 | 0.31296 | 0.39357 | 0.38101 | 0.38728 | 0.33960 | 0.27640 |
| | $\beta$ | 2.05470 | 0.39719 | 0.33233 | 0.34994 | 0.34109 | (2.66385, 3.4030) | (0.859712, 1.20527) |
| | | 0.89589 | 0.39389 | 0.36030 | 0.36972 | 0.36502 | 0.73916 | 0.34556 |
| 500 | $\alpha$ | 0.03653 | 0.00599 | 0.01862 | 0.01956 | 0.01909 | (0.18592, 0.23185) | (0.22443, 0.30586) |
| (250,125) | | 0.47779 | 0.19357 | 0.34117 | 0.34961 | 0.34538 | 0.04593 | 0.08143 |
| (250,125) | $\theta$ | 0.82983 | 0.02052 | 0.06928 | 0.06256 | 0.06588 | (1.47727, 1.84463) | (0.86201, 1.18052) |
| | | 1.21460 | 0.19101 | 0.35096 | 0.33348 | 0.34222 | 0.36737 | 0.31851 |
| | $\beta$ | 5.38390 | 0.41235 | 0.38116 | 0.40734 | 0.39417 | (3.40241, 4.43823) | (0.79691, 1.19376) |
| | | 1.45020 | 0.40134 | 0.38586 | 0.39889 | 0.39239 | 1.03582 | 0.39685 |

3. The MLEs and the Bayes estimates for case (1) ($\alpha = 0.1, \theta = 2, \beta = 2$) have good statistical properties for the MSEs and RABs in most of the cases considered see Table 1.
4. In general, under SEL and GE loss function when ($c$ = 3) the Bayes estimates of $\alpha$ provides the smallest MSEs and RABs as compared with MLE estimates see Table 3.
5. In general, the Bayes estimates of $\theta$ under GE loss function when ($c$ = –1,3) have the smallest MSEs and RABs as discussed with MLEs and SEL loss function estimates see Table 3.
6. In general, the Bayes estimates of $\beta$ have the smallest MSEs and RABs under SEL loss function as compared with GE loss function and MLEs estimates see Table 3.
7. For case (2)($\alpha = 1.5, \theta = 4, \beta = 2$), the approximate CIs of $\alpha$, $\theta$ and $\beta$ yield a good results than MCMC credible CIs for the length of CIs see Table 4.
8. It can also be seen that for case (2)($\alpha = 2, \theta = 4, \beta = 1.2$), the MLEs of $\alpha$, $\theta$, and $\beta$ have a good outcomes for the MSEs and RABs in most of the cases considered see Table 4.
9. In general, for the sample size ($n$ = 60,100) the SEL function of $\alpha$ have the smallest MSEs and RABs as discussed with MLE and GE loss function estimates, whereas, for the sample size ($n$ = 150,300,500) the MLE of $\alpha$ have the smallest MSEs and RABs as compared with SEL and GE loss function estimates see Table 2.
10. Also, the SEL and GE loss function of $\theta$ have the smallest MSEs and RABs as compared with MLE see Table 4.
11. For case (3) ($\alpha = 0.4, \theta = 0.75, \beta = 1.6$), the MCMC credible CIs of the three parameters yield a good estimate than approximate CIs. Also, the MCMC credible intervals give lower lengths for the small and large sample sizes see Table 5.

## 6. CONCLUSION

This paper provides the problem of estimate the EW distribution under partially constant-stress ALTs with multiple Type-II censored data. The maximum likelihood and Bayes methods are utilized to estimation the model parameters. Additionally, the two-sided confidence limit and credible CIs are computed. Bayesian estimation problem can be extended to include the two, informative and non-informative priors with help of one of the important techniques called MCMC technique to obtained the empirical posterior distribution which used to approximate Bayes estimates and credible confidence intervals.

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