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Short Paper

Properties and Embeddings of Interconnection Networks Based on the Hexcube

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A new class of interconnection networks called the *hexcube* is proposed. The hexcube is similar to the base-6 generalized hypercube in structure but has a simpler interconnection scheme. The present work shows that the hexcube is vertex symmetric and possesses topological properties similar to those of the hypercube. This implies that the costs of building parallel computers using the hexcube and using the binary hypercube are similar, and are much lower than those incurred using the based-6 generalized hypercube. A one-port broadcasting algorithm for the hexcube is proposed. New results for embeddings using the hexcube as the host topology are also presented. First, a reflected Gray code-like method for finding Hamiltonian cycles is developed. Second, algorithms for all two-dimensional mesh embedding with unit expansion and a dilation of no more than two are developed. Third, it is shown that a relatively large binary hypercube can be embedded into a hexcube with a dilation of no more than three and with almost optimal expansion.

Keywords: interconnection networks, hexcube, hypercube, one-port broadcasting, Hamiltonian cycles, mesh embeddings, binary hypercube embeddings

1. INTRODUCTION

Rapid advances in microprocessor and network technologies have opened up the possibility of building high performance, massively parallel multiprocessor systems. In such a system, thousands of processors are connected with an interconnection network. Several topologies have been proposed as interconnection networks for multiprocessor systems [1-13]. Among them, a class of networks based on the *hypercube* has received most attention from researchers [4, 14]. *Symmetry, good topological properties*, and *excellent embeddability* [14, 15] are the main reasons for the hypercube's popularity. The main advantage of the symmetric network is in the development and porting of parallel algo-

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rithms for the host topology. Specifically, since a symmetric network reveals the same topology when viewed from any node, parallel algorithms can be developed on any single node and then ported to the host multiprocessor system. Topological properties that are important to an interconnection network include the *degree*, *order*, *number of edges*, *diameter*, and the *average distance* between any two nodes. The degree and number of edges directly affect the number of communication ports for each processor and the total number of communication links, respectively, which jointly account for most of the communication hardware cost; the diameter determines the worst case communication delay, and the average distance directly affects network congestion [16]. Finally, embeddability of one interconnection network into another means that all parallel and distributed algorithms developed for the former can be readily ported to the latter. In addition, good embeddability is also a requirement for economical mapping of the task/data flow graphs of parallel algorithms into the interconnection topology.

In this paper, a class of interconnection networks called the *hexcube* is proposed. The hexcube is similar to the *base-6 generalized hypercube* in structure but has a simpler interconnection scheme. It is shown that the hexcube is vertex symmetric. Moreover, hexcubes possess topological properties that are very similar to those of binary hypercubes. Thus, the costs of hardware and communication and the cost of parallel algorithm development for the hexcube are to a large degree comparable to those for the binary hypercube. In order to demonstrate the flexibility of the hexcube, we further present several algorithms, including algorithms for one-port broadcasting and several embeddings. In one-port broadcasting, it is shown that the maximum transmission delay is no more than $3n$, where n is the dimension of the hexcube. New results for embedding using the hexcube are also presented. First, a reflected Gray code-like method for finding a Hamiltonian cycle for the hexcube is developed. Second, algorithms for all two-dimensional mesh embeddings with unit expansion for the hexcube are developed. The dilation of these embeddings is no more than two. Third, it is shown that a relatively large binary hypercube can be embedded into a hexcube with a dilation of no more than three and with almost optimal expansion. The embeddability of the binary hypercube into the hexcube is significant because a large inventory of parallel algorithms for high performance computers based on the binary hypercube already exists. From these results, the hexcube can be considered as a viable class of interconnection networks for building large scale multiprocessor systems.

The organization of this paper is as follows. In section 2, the definition of the hexcube and its properties are given. In section 3, results on average distance and a one-port broadcasting algorithm are presented. Section 4 develops the embedding algorithms for the hexcube, including embeddings for the Hamiltonian cycle, various two-dimensional meshes, and the binary hypercube. Section 5 summarizes the results.

2. DEFINITIONS AND PROPERTIES OF THE HEXCUBE

Let $< n > = \{0, 1, \cdots, n-1\}$. Define the hexcube $HC_n = (V, E)$ of dimension n as $V = \{ x \mid x = x_1x_2 \cdots x_n$, where $x_i \in < 6 >\}$, and $E = \{(x, y) \mid x, y \in V$, and there exists $1 \leq j \leq n$ such that $y_j = (x_j \pm 1) \bmod 6$, and $x_i = y_i$ for all $1 \leq i \neq j \leq n\}$. Examples of HC_n for $n = 1, 2$ are given in Fig. 1. Clearly, HC_n is an undirected graph and can be built recursively by using six copies of HC_{n-1}. Furthermore, it is readily verified that HC_n is a regular graph with a degree of $2n$ and an order of 6^n .

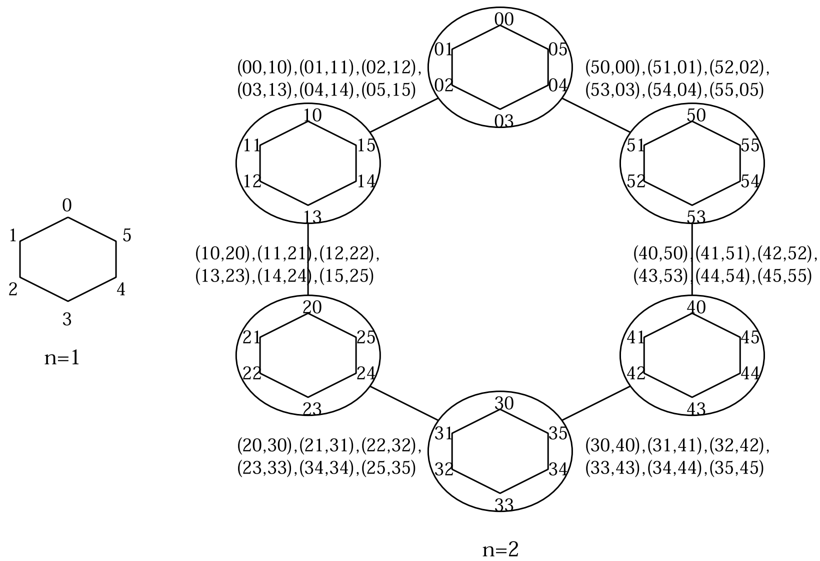

Fig. 1. Examples of hexcubes: HC_1 and HC_2.

In developing an interconnection network as the architecture for high performance computers, investigation of symmetric properties for the underlying graph of the network is necessary [1-3, 15]. The hexcube is vertex symmetric. To see this, note that HC_n is isomorphic to the Cayley graph based on the permutation group with the generator set $\{(3i-2\ 3i-1) | 1 \le i \le n\} \cup \{(3i-2\ 3i) | 1 \le i \le n\}$, where $(a\ b)$ is the traditional cycle structure representation for permutation [15]. Examples of the above Cayley graph for $n = 1$ and $n = 2$ are given in Fig. 2. The isomorphism ϕ between the two graphs can be easily seen by observing the following coding scheme:

$$\phi(x_i) = \begin{cases} 3i-2 \quad 3i-1 \quad 3i & \text{when} \quad x_i = 0 \\ 3i-1 \quad 3i-2 \quad 3i & \text{when} \quad x_i = 1 \\ 3i-1 \quad 3i \quad 3i-2 & \text{when} \quad x_i = 2 \\ 3i \quad 3i-1 \quad 3i-2 & \text{when} \quad x_i = 3 \\ 3i \quad 3i-2 \quad 3i-1 & \text{when} \quad x_i = 4 \\ 3i-2 \quad 3i \quad 3i-1 & \text{when} \quad x_i = 5 \end{cases}$$

for $1 \le i \le n$. As an example, note that node 12 in Fig. 1 is mapped to node 213564 in Fig. 2. The fact that the Cayley graph is vertex symmetric [15] establishes the next lemma.

Lemma 2.1. The hexcube is vertex symmetric.

To facilitate subsequent development in this paper, we also need an algorithm for routing packets between a pair of nodes in HC_n. Let $x = x_1x_2 \cdots x_n$ be the source node and $y = y_1y_2 \cdots y_n$ be the destination node. In deciding the next node $x' = x_1'x_2' \cdots x_n'$ in the route, we find the first i such that $x_i \ne y_i$ for some $1 \le i \le n$. The process is then repeated from the newly reached node, and so on, until the destination node is reached. The process is summarized in Fig. 3.

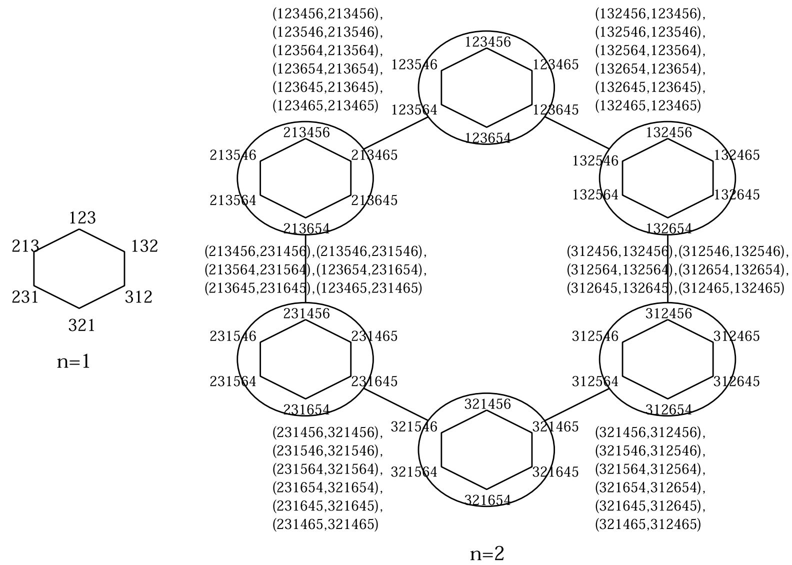

Fig. 2. Examples of Cayley graphs: EHC_1 and EHC_2.

Let $x = x_1x_2...x_n$ be the source node;
Let $y = y_1y_2...y_n$ be the destination node;
$i = 1$;
while $i \leq n$ do
    while $e = x_i - y_i \neq 0$ do
        if ($|e| \leq 3$) then
            if $e > 0$ then $x_i = (x_i - 1) \bmod 6$;
            else $x_i = (x_i + 1) \bmod 6$;
        else
            if $e > 0$ then $x_i = (x_i + 1) \bmod 6$;
            else $x_i = (x_i - 1) \bmod 6$;
        end_while; /* $x_i = y_i$*/
        $i = i + 1$;
end_while;/* $x = y$*/

Fig. 3. A shortest path routing algorithm for HC_n.

As an example, let $x = 0153$ and $y = 2511$. Then, the routing path from x to y is as follows:

$$x = 0153 \to 1153 \to 2153 \to 2053 \to 2553 \to 2503 \to 2513 \to 2512 \to 2511 = y.$$

It can be verified that the above routing algorithm generates a shortest path between the source and destination nodes. Furthermore, define m_i as follows:

$$m_i = \begin{cases} |x_i - y_i|, & \text{if } |x_i - y_i| \leq 3, \\ 6 - |x_i - y_i|, & \text{if } |x_i - y_i| > 3, \end{cases} \tag{1}$$

for $1 \le i \le n$. Then, the distance $d(x, y)$ between x and y can be calculated as follows:

$$d(x,y) = \sum_{i=1}^{n} m_i. \tag{2}$$

From equation (1), it is obvious that $m_i \le 3$ for all $1 \le i \le n$. Additionally, it is clear that the longest distance between any two nodes of HC_n is $3n$ when the difference of the corresponding pair of digits, say x_i and y_i, for all $1 \le i \le n$, is three. Table 1 provides a summary in which comparisons among the hexcube, the binary hypercube and the base-6 generalized hypercube are given.

In the treatment of mesh and hypercube embeddings, we also need the following definition [17]. Given the graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, defind their *product graph* $G = G_1 \times G_2$ such that its vertex set $V = \{[x, y] \mid x \in V_1, y \in V_2\}$ and edge set $E = \{([x, y], [x, y']) \mid x \in V_1, (y, y') \in E_2\} \cup \{([x, y], [x', y]) \mid y \in V_2, (x, x') \in E_1\}$.

Table 1. Comparisons among the hexcube, the binary hypercube and the base-6 hypercube.

	dimension	degree	order	diameter	vertex symmetry
hexcube	n	$2n$	6^n	$3n$	yes
binary hypercube	$2.45n$	$2.45n$	6^n	$2.45n$	yes
base-6 generalized hypercube	n	$5n$	6^n	n	yes

3. AVERAGE DISTANCE AND ONE-PORT BROADCASTING

In this section, the average distance of the hexcube is computed. A method for one-port broadcasting, a fundamental communication scheme which is necessary for many parallel computing applications, is also proposed.

3.1 Average Distance

The average distance directly determines the communication costs. Specifically, since there is only one link between two adjacent nodes, contention occurs when two different messages compete for that link; moreover, it has been shown that network contention increases proportionally as the average distance increases [16].

Without loss of generality, let $T(n)$ be the total distance between *identity* vertex $I = 0^n = 00 \cdots 0$ and all the other vertices in HC_n. Let HC_n^i denote the subgraph of HC_n with the nth digit equal to i. Note that HC_n^i is isomorphic to HC_{n-1} and has 6^{n-1} vertices. In HC_n^i, the total distance from vertex $00 \cdots i$ to all the other vertices is $T(n-1)$. It follows that the total distance from the identity vertex I to all vertices in HC_n^i is $T(n-1) + i \cdot 6^{n-1}$ for $0 \le i \le 3$ and $T(n-1) + (6-i) \cdot 6^{n-1}$ for $4 \le i \le 5$. Thus, the total distance between I and all the other vertices in HC_n is

$$T(n) = T(n-1) + 2(T(n-1) + 6^{n-1}) + 2(T(n-1) + 2 \cdot 6^{n-1}) + (T(n-1) + 3 \cdot 6^{n-1}). \qquad (3)$$

From equation (3), it can be verified that $T(n) = 9 \times 6^{n-1}n$. Since the hexcube is vertex symmetric, the average distance of HC_n is $\frac{T(n)}{6^n}$, and we have the following lemma.

Lemma 3.1. The average distance of HC_n is $1.5n$.

Notice that the average distance of the hexcube is one half of its diameter; the same relationship holds for the binary hypercube.

3.2 One-port Broadcasting

In the rest of this section, a one-port broadcasting algorithm for the hexcube is proposed. A broadcasting is said to be a one-port broadcasting if each node of the network can only send a packet to one of its neighbors in a communication cycle. Since the hexcube is vertex symmetric, it suffices to consider the broadcasting algorithm for node I. For the other cases, the broadcasting algorithm can be easily achieved by using the corresponding automorphism. The broadcasting algorithm is shown in Fig. 4. Since the transmission pattern (Fig. 5) that is used in each of the n parallel steps of the broadcasting algorithm has a height of three, we have the following lemma.

```
Let τ be the packet to be broadcast from I in HC_n;
for i = 1 to n do in parallel
        for each node x which has already received τ do
              sends τ to the five nodes whose ith digit
              is different from x using the transmission
              pattern given in Fig. 5;
```

Fig. 4. A one-port broadcasting algorithm for HC_n.

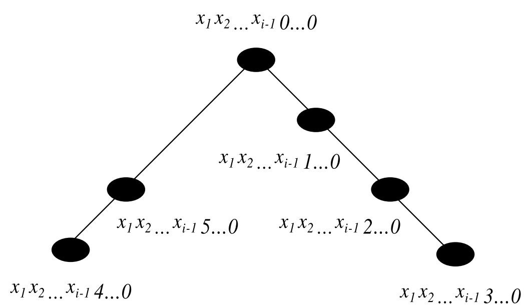

Fig. 5. The transmission pattern used by node $x_1x_2 \cdots x_{i-1}\, 0 \cdots 0$ in step i of the the one-port broadcasting algorithm.

Lemma 3.2. The total delay of one-port broadcasting in HC_n is $3n$.

Notice that according to the transmission pattern, each node in HC_n will receive the broadcast packet exactly once; Fig. 6 shows the case for HC_3, where the nodes that receive the packet in each parallel step are grouped and marked with $i = 1$, $i = 2$, and $i = 3$, respectively.

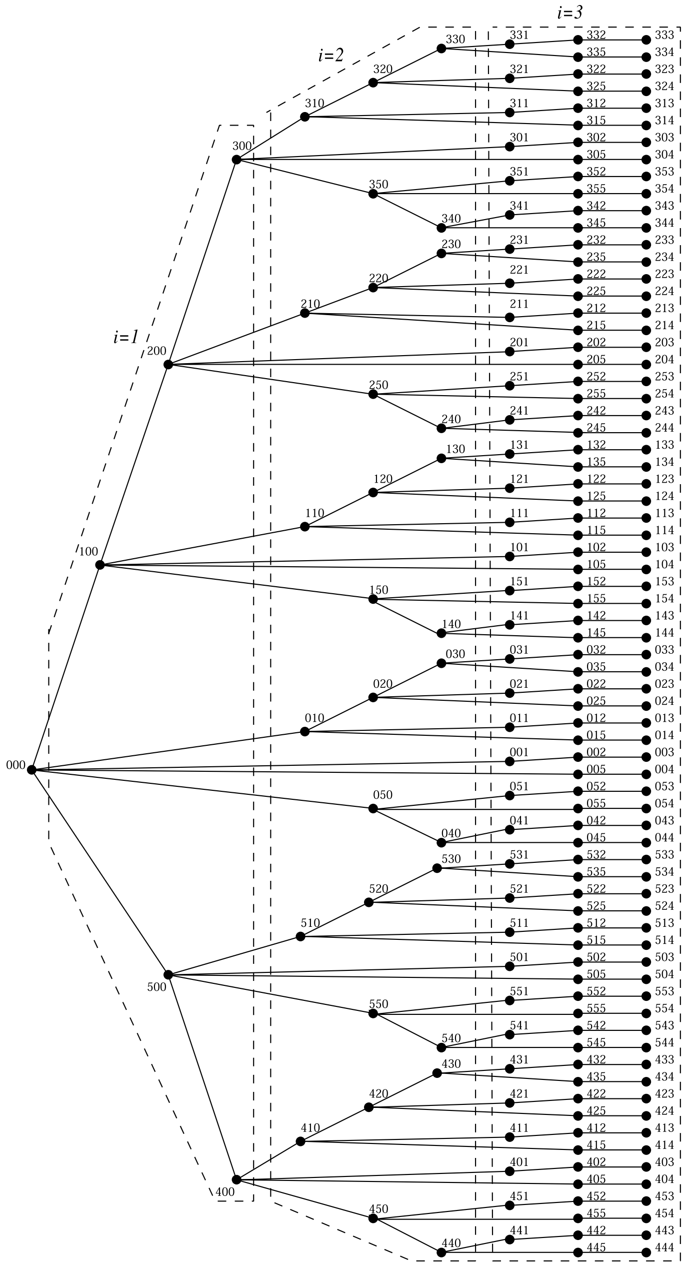

Fig. 6. One-port broadcasting in HC_n, where $n = 3$.

4. EMBEDDINGS FOR THE HEXCUBE

In this section, we present various embedding algorithms using the hexcube as the host topology. By embedding we mean that, given a guest graph $G(V_G, E_G)$ and a host graph $H(V_H, E_H)$, there exists a mapping function α such that $\alpha(u) = v$ where $u \in V_G$ and $v \in V_H$. Two parameters used in the evaluation of an embedding α are *dilation* and *expansion*, defined, respectively, as $\max(d(\alpha(s), \alpha(t)))$ for all $s, t \in V_G$ and $\frac{|V_H|}{|V_G|}$, where $d(x, y)$ is the distance between vertices x and y.

4.1 Hamiltonian Cycle

The existence of a Hamiltonian cycle in the underlying graph of a given interconnection network is critical to the implementation of some parallel algorithms [14]. Let $G(n)$ denote a sequence of all n-digit base-6 words. Define $G(1) = \{0, 1, 2, 3, 4, 5\}$ and $G(n) = \{G_0(n), G_1(n), \cdots, G_{6^n-1}(n)\}$, where $G_i(n)$ is called the *encoding* of integer i for $0 \le i \le 6^n - 1$. With this definition, the sequence $G(n + 1)$ can be derived recursively as follows:

$$\begin{aligned} G(n+1) = \{&0|G_0(n), 0|G_1(n), \ldots, 0|G_{6^n-1}(n), \\ &1|G_{6^n-1}(n), 1|G_{6^n-2}(n), \ldots 1|G_0(n), \\ &2|G_0(n), 2|G_1(n), \ldots, 2|G_{6^n-1}(n), \\ &3|G_{6^n-1}(n), 3|G_{6^n-2}(n), \ldots, 3|G_0(n), \\ &4|G_0(n), 4|G_1(n), \ldots, 4|G_{6^n-1}(n), \\ &5|G_{6^n-1}(n), 5|G_{6^n-2}(n), \ldots, 5|G_0(n)\}, \end{aligned} \tag{4}$$

where " | " denotes *concatenation* of the words. It can be easily verified that any two consecutive words given in equation (4) are adjacent in HC_n, including the first and the last two words. Thus, we have the following lemma.

Lemma 4.1.1. $G(n)$ sequence is a Hamiltonian cycle for HC_n.

An example of $G(2)$ is listed in Table 2. Note that our definition of $G(n)$ is similar to that for the reflected Gray code for the hypercubes.

Table 2. A Hamiltonian cycle of HC_2.

00	15	20	35	40	55
01	14	21	34	41	54
02	13	22	33	42	53
03	12	23	32	43	52
04	11	24	31	44	51
05	10	25	30	45	50

4.2 Mesh Embeddings

Meshes form an important class of networks that is useful in an area like image processing. In this subsection, new results for the embedding of all meshes into the hexcube with unit expansion are presented. Note that some of these meshes are very difficult to embed into the binary hypercube. We begin our discussion with the following lemmas and corollary.

Lemma 4.2.1. $HC_{t_1} \times HC_{t_2} = HC_{t_1+t_2}$.

Proof: Let $[x, y]$ and $[u, v]$ be two vertices of $HC_{t_1} \times HC_{t_2}$. It is obvious that $x \mid y$ and $u \mid v$ are two vertices of $HC_{t_1+t_2}$. Since $[x, y]$ and $[u, v]$ are adjacent if and only if either $x = u$ and (y, v) is an edge of HC_{t_2} or $y = v$ and (x, u) is an edge of HC_{t_1}, it is obvious that $(x \mid y, u \mid v)$ is an edge of $HC_{t_1+t_2}$. Following the same argument, it can also be seen that there exists an isomorphism to map $HC_{t_1+t_2}$ into $HC_{t_1} \times HC_{t_2}$. □

The next corollary is an immediate result of Lemma 4.2.1.

Corollary 4.2.2. $HC_n = HC_{t_1} \times HC_{t_2} \times \cdots \times HC_{t_r}$ where $1 \leq r$, t_i is a positive integer for $1 \leq i \leq r$ and $n = \sum_{i=1}^{r} t_i$.

The next result shows that power-of-six meshes can be embedded into a corresponding hexcube with unit dilation and unit expansion.

Lemma 4.2.3. A $6^{k_1} \times 6^{k_2}$ mesh can be embedded into $HC_{k_1+k_2}$ with unit dilation and unit expansion.

Proof: Label the first dimension and second dimension of the mesh with $G(k_1)$ and $G(k_2)$ sequences as defined in section 4.1, respectively. Consider any node of the mesh labeled by (u, v) such that u and v are the coordinates, where $u \in G(k_1)$ and $v \in G(k_2)$. It is obvious that $u \mid v$ is a node of $HC_{k_1+k_2}$. Since $G(k_1)$ and $G(k_2)$ are Hamiltonian cycles of HC_{k_1} and HC_{k_2}, respectively, it can be verified easily that for all neighboring nodes of (u, v) in the mesh, say (u_1, v_1), (u_2, v_2), (u_3, v_3) and (u_4, v_4), the nodes $u_1 \mid v_1$, $u_2 \mid v_2$, $u_3 \mid v_3$ and $u_4 \mid v_4$ in $HC_{k_1+k_2}$ are adjacent to $u \mid v$ and hence the lemma. □

Fig. 7 gives an example of a 6×6 mesh embedding in HC_2 accomplished by applying Lemma 4.2.3.

50	51	52	53	54	55
40	41	42	43	44	45
30	31	32	33	34	35
20	21	22	23	24	25
10	11	12	13	14	15
00	01	02	03	04	05

Fig. 7. An example of embedding 6×6 mesh into HC_2.

Let $G = G_1 \times G_2 \times \cdots \times G_r$, $H = H_1 \times H_2 \times \cdots \times H_r$, and l_i be the dilation of embedding G_i into H_i for $1 \le i \le r$. The following lemma is due to [17].

Lemma 4.2.4. Product graph G can be embedded into H with dilation l, where

$$l = \max\{l_i \mid 1 \le i \le r\}.$$

Using the techniques of mesh decomposition given in [17], we can come up with the lemma shown below.

Lemma 4.2.5. A $(l_{11} \cdot l_{12} \cdots l_{1k}) \times (l_{21} \cdot l_{22} \cdots l_{2k})$ mesh is a subgraph of $M_{l_{11} \cdot l_{21}} \times M_{l_{12} \cdot l_{22}} \times \cdots \times M_{l_{1k} \cdot l_{2k}}$, where $M_{i \cdot j}$ is an $i \times j$ mesh.

One of the fundamental results of this section is given in the following lemma.

Lemma 4.2.6. A $2^k \times 3^k$ mesh can be embedded into HC_k with a dilation of two and unit expansion.

Proof: By observing Fig. 7, it is obvious that a 2×3 mesh can be embedded into HC_1 with a dilation of two. Following Lemma 4.2.5, the $2^k \times 3^k$ mesh is a subgraph of $M_1 \times M_2 \times \cdots \times M_k$, where M_i is a 2×3 mesh for $1 \le i \le k$. By applying Corollary 4.2.2 and Lemma 4.2.4, $M_1 \times M_2 \times \cdots M_k$ can be embedded into the product graph of k copies of HC_1 with a dilation of two. □

00	01	02	12	11	10	20	21	22
05	03	04	14	13	15	25	23	24
55	53	54	34	33	35	45	43	44
50	51	52	32	31	30	40	41	42

Fig. 8. An example of embedding a 4×9 mesh into HC_2.

Fig. 8 gives an example of a 4×9 mesh embedded in HC_2 by applying Lemma 4.2.6.

The following theorem is an immediate result of the above lemmas and corollary.

Theorem 4.2.7. Let t_1, t_2, t_3, and t_4 be any four positive integers. A $(2^{t_1} \cdot 3^{t_2}) \times (2^{n-t_1} \cdot 3^{n-t_2})$ mesh M can be embedded into HC_n with a dilation of two and unit expansion.

Proof: Without loss of generality, let $t_1 < t_2$. Mesh M can be viewed as a $(6^{t_1} \cdot 3^{t_2-t_1}) \times (6^{n-t_2} \cdot 2^{t_2-t_1})$ mesh. By Lemma 4.2.5, mesh M is a subgraph of the product of the two meshes $(6^{t_1} \times 6^{n-t_2})$ and $(3^{t_2-t_1} \times 2^{t_2-t_1})$. By Lemmas 4.2.3, 4.2.4 and 4.2.6, mesh $(6^{t_1} \times 6^{n-t_2})$ and mesh $(3^{t_2-t_1} \times 2^{t_2-t_1})$ can be embedded into $HC_{n+t_1-t_2}$ and $HC_{t_2-t_1}$ with a dilation of one and two, respectively. Thus, the theorem follows from Lemmas 4.2.1 and 4.2.4. □

Note that since 2 and 3 are the only prime factors of the number 6^n, Theorem 4.2.7 has covered all the possible unit expansion mesh embedding. Fig. 9 shows an embedding of a 12×18 mesh into HC_3.

500	501	502	512	511	510	520	521	522	532	531	530	540	541	542	552	551	550
505	503	504	514	513	515	525	523	524	534	533	535	545	543	544	554	553	555
405	403	404	414	413	415	425	423	424	434	433	435	445	443	444	454	453	455
400	401	402	412	411	410	420	421	422	432	431	430	440	441	442	452	451	450
300	301	302	312	311	310	320	321	322	332	331	330	340	341	342	352	351	350
305	303	304	314	313	315	325	323	324	334	333	335	345	343	344	354	353	355
205	203	204	214	213	215	225	223	224	234	233	235	245	243	244	254	253	255
200	201	202	212	211	210	220	221	222	232	231	230	240	241	242	252	251	250
100	101	102	112	111	110	120	121	122	132	131	130	140	141	142	152	151	150
105	103	104	114	113	115	125	123	124	134	133	135	145	143	144	154	153	155
005	003	004	014	013	015	025	023	024	034	033	035	045	043	044	054	053	055
000	001	002	012	011	010	020	021	022	032	031	030	040	041	042	052	051	050

Fig. 9. An example of embedding a 12×18 mesh into HC_3.

4.3 Binary Hypercube Embeddings

In this section, following the same idea used in the graph decomposition approach proposed previously, we shall investigate embeddings of the binary hypercube in the hexcube. To begin with, we give a well-known result due to [17].

Lemma 4.3.1. Let $r = r_1 + r_2 + \cdots r_k$, where r_i is a positive integer for $1 \le i \le k$. Then, the r-dimensional hypercube Q_r is isomorphic to the product graph of the hypercubes Q_{r_i} for all $1 \le i \le k$.

Since the orders of the binary hypercube and the hexcube are different, we consider only the *expansion optimal* embeddings. An embedding is said to be expansion optimal if we can find a smallest hexcube into which the given binary hypercube can be embedded. The possible expansion optimal embeddings for Q_1, Q_2, Q_3, Q_4, and Q_5 are listed in Table 3. Note that the dilation of these embeddings is at most three. For a given n dimensional hypercube Q_n, we first decompose it into $Q_n = Q_5^{\lfloor \frac{n}{5} \rfloor} \times Q_{n(\text{mod } 5)}$, where $Q_5^{\lfloor \frac{n}{5} \rfloor}$ represents the product of $\lfloor \frac{n}{5} \rfloor$ copies of Q_5. Since each Q_5 can be embedded into HC_2 with a dilation of three and $Q_{n\,(\text{mod } 5)}$ can be embedded into either HC_1 or HC_2 from Corollary 4.2.2 and Lemma 4.2.4, we obtain the following theorem.

Theorem 4.3.2. Q_n can be embedded into

(1) $HC_{\frac{2n}{5}}$ with dilation three and expansion $2^n / 6^{\frac{2n}{5}}$ if n (mod 5) = 0;

(2) $HC_{2\lfloor \frac{n}{5} \rfloor + 1}$ with dilation three and expansion $2^n / 6^{2\lfloor \frac{n}{5} \rfloor + 1}$ if n (mod 5) = 1 or 2; or

(3) $HC_{2\lfloor \frac{n}{5} \rfloor + 2}$ with dilation three and expansion $2^n / 6^{2\lfloor \frac{n}{5} \rfloor + 2}$ if n (mod 5) = 3 or 4.

Table 3. The possible expansion optimal embeddings for Q_1, Q_2, Q_3, Q_4, and Q_5.

(a) Embedding Q_1 in HC_1 with dilation one.

Q_1	HC_1
0	0
1	1

(b) Embedding Q_2 in HC_1 with dilation three.

Q_2	HC_1
00	1
01	2
10	5
11	4

(c) Embedding Q_3 in HC_2 with dilation two.

Q_3	HC_2	Q_3	HC_2
000	01	100	11
001	02	101	12
010	05	110	15
011	04	111	14

(d) Embedding Q_4 in HC_2 with dilation two.

Q_4	HC_2	Q_4	HC_2	Q_4	HC_2	Q_4	HC_2
0000	01	0100	11	1000	51	1100	31
0001	02	0101	12	1001	52	1101	32
0010	05	0110	15	1010	55	1110	35
0011	04	0111	14	1011	54	1111	34

(e) Embedding Q_5 in HC_2 with dilation three.

Q_5	HC_2	Q_5	HC_2	Q_5	HC_2	Q_5	HC_2
00000	01	01000	21	10000	51	11000	10
00001	02	01001	22	10001	52	11001	13
00010	05	01010	25	10010	55	11010	00
00011	04	01011	24	10011	54	11011	03
000100	11	01100	41	10100	31	11100	30
00101	12	01101	42	10101	32	11101	33
00110	15	01110	45	10110	35	11110	40
00111	14	01111	44	10111	34	11111	43

By means of simple calculations, it can be seen that except for n = 18, 23 and 28, the proposed embedding is almost expansion optimal and has a dilation of three when $n \leq 30$.

5. CONCLUSIONS

The hexcube is similar to the base-6 hypercube in structure but has a simpler interconnection scheme. In this paper, we have shown that the hexcube is vertex symmetric and possesses topological properties similar to those of the binary hypercubes. For embeddings, first, a reflected Gray-code like method for producing a Hamiltonian cycle for the hexcube has been developed. Second, algorithms for all two-dimensional mesh embeddings with unit expansion for the hexcube have been developed. The dilation of these embeddings is no more than two. Third, it has been shown that a relatively large binary hypercube can be embedded into a hexcube with a dilation of no more than three and almost optimal expansion. From these results, the hexcube can be considered to be a viable type of interconnection network for building large scale multiprocessor systems.

Theoretically, the idea of the hexcube can be extended to base-*n* structures. We are currently working on some possible extensions.

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