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Joint Type-I Generalized Hybrid Censoring for Estimation Two Weibull Distributions

ALI ALGARNI[1], ABDULLAH M. ALMARASHI[1]
AND G. A. ABD-ELMOUGOD[2]
[1]*Statistics Department*
King Abdulaziz University
Jeddah, 21589 Saudi Arabia
[2]*Mathematics Department*
Damanhour University
Damanhour, 22511 Egypt
E-mail: ahalgarni@kau.edu.sa; almarashi.research@gmail.com; gamalamin@sci.dmu.edu.eg

Products come from different lines with the same facility are tested under comparative life tests which known with the jointly censoring scheme. In this paper, two sets of products under the same facility have Weibull lifetime distributions are selected to test under Type-I generalized hybrid censoring scheme (GHCS). The observed censoring data are used to build the maximum likelihood (ML) estimators as well as approximate confidence intervals for the model parameters. Also, Bayes estimators with the help of MCMC methods are discussed. The analysis of simulated data set with Monte Carlo simulation study is used to illustrate and compare the theoretical results. Finally, a brief comment is summarized in concluding section.

Keywords: joint Type-I generalized hybrid censoring, Weibull distributions, maximum likelihood estimation, Bayesian estimation, MCMC

1. INTRODUCTION

The data obtained from the life tests experiments, may be complete or censored. When the exact failure time of all units in the experiment can be obtained then, the data called complete data. But, under consideration time and cost when failure time of some units don't observe until the end of the experiment, censoring data is applied. The common censoring scheme in life testing experiments is called Type-I and Type-II censoring. In Type-I censoring scheme, the test time is constant and the number of failures is random may be zero see, [1] but in Type-II censoring scheme, number of failures is constant and the test time is random may be very large. Hybrid censoring scheme (HCS) is a mixture of Type-I and Type-II censoring schemes which at the prior of the experiment the fixed integer m and fixed time τ are determined. The experiment is terminated when the number m of failures or time τ has been reached. In Type-I HCS, the experiment is terminated at min (T_m, τ), see in more detail [2, 3]. In Type-II HCS, the experiment is terminated at max (T_m, τ), see in more detail [4]. In the two types of censoring, Type-I HCS and Type-II HCS number of failure units may be very few or even no failures or experiment has

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long period of time, respectively see [5]. Generalized hybrid censoring scheme (GHCS) is applied to overcome of this problem see [6].

Type-I GHCS scheme described as follows, suppose n units are put on a life test experiment and two fixed integer k, m such that $1 \leq k < m \leq n$ and time $\tau \in (0, \infty)$ is determined. If $T_k < \tau$ the experiment is terminated at min (T_m, τ) but if $T_k > \tau$, the experiment is terminated at T_k. Therefore, in Type-I GHCS experiment satisfies the minimum number k of failures. Then, the data come from Type-I GHCS were summarized as

$$\underline{t} = \begin{cases} \text{Case 1: If } t_{k;n} > \tau, \text{ then } \underline{t} = (t_{1;n} < t_{2;n} < \ldots < t_{k;n}), \\ \text{Case 2: If } t_{k;n} < \tau, \text{ then } \underline{t} = (t_{1;n} < t_{2;n} < \ldots < t_{k;n} < \ldots < t_{r;n}) \text{ at } t_{m;n} > \tau, \\ \underline{t} = (t_{1;n} < t_{1;n} < \ldots < t_{m;n}) \text{ at } t_{m;n} < \tau, \end{cases} \tag{1}$$

where different cases of censoring with Type-I GHCS are summarized in Fig. 1 below,

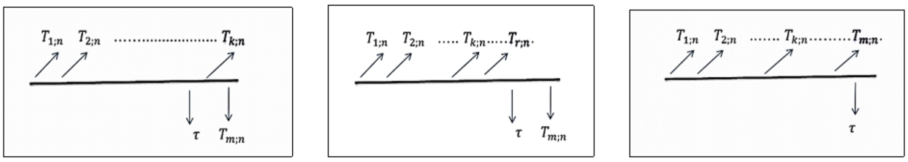

Fig. 1. Different cases of Type-I GHCS.

then the joint density function of Type-I GHCS given the parameters vector θ is given by

$$f_{1,2,\ldots,m}(\underline{t}|\theta) = \frac{n!\,(1-F(C))^{n-D}}{(n-D)!} \prod_{i=1}^{D} f(t_i), \tag{2}$$

where

$$\begin{cases} \text{Case 1:} \quad D = k \text{ and } C = t_k \text{ at } t_{k;n} > \tau, \\ \text{Case 2:} \quad D = r,\ k < r < m \text{ and } C = \tau \text{ at } t_{m;n} > \tau, \\ \qquad\qquad D = m \text{ and } C = t_m \text{ at } t_{m;n} < \tau. \end{cases} \tag{3}$$

Studying the reliability of manufactured products to determine and measure the relative merits of two life products through the competing duration has considerable in the last view years. For more precise, we consider a manufactured products come from the two different lines Φ_1 and Φ_2 are putted under the same conditions. The two independent samples of size M and N are choosed from Φ_1 and Φ_2, respectively, to placed together under test. Then, the experimenter may be terminated for consideration of cost and time after fixed number of failures occur. The two failure times and it is types will be recorded. Different author discussed this type of censoring scheme see [7, 8]. Also, for the comparing of the exact likelihood inference with bootstrap technique see [9]. And for progressive Type-II censoring see [10, 11] . Recently, for the two Rayleigh lifetime distributions see [12], for Accelerate life test of Rayleigh life time distribution see [13] and for compound Rayleigh lifetime distributions see [14].

A short development times for products in present time make some time limitations over reliability tests, which impulse that joint censoring scheme need some modification which save time and give a suitable number of failure which serve statistical inference. Therefore, Type-I GHCS introduce a new scheme where save time and minimum number that needing in statistical inference. Then, our objective in this paper present inferences for important lifetime Weibull distribution under Type-I GHCS scheme, then problem of parameters estimation of two weibull distributions when Type-I GHCS samples is available. Then, maximum likelihood as well as Bayes estimation are used to present the estimation of unknown model parameters. Different estimators are discussed and compared through simulation experiments and numerical example based on Type-I GHCS.

This paper is summarized as follows: The model formulation and main concepts are discussed in Section 2. The maximum likelihood, the point and approximate intervals estimators for the unknown parameters are derived in Section 3. Bayes estimators under the concepts of MCMC method for point and credible interval estimation are presented in Section 4. The analysis of simulated data sets exposed in Section 5. Reported some of numerical results are discussed through simulation study in Section 6. Finally, a brief comments about the obtaining numerical results are constructed in Section 7.

2. MODEL

Suppose we have two line of production, say Φ_1 and Φ_2 has produce the same product under the same facility. Let two independent samples of sizes M and N are selected from the lines Φ_1 and Φ_2 which has independent and identical distributed (i.i.d) lifetimes $X_1, X_2, ..., X_M$ and $Y_1, Y_2, ..., Y_N$, respectively. The two lifetime samples has a populations with probability density functions (PDFs) and cumulative distribution functions (CDFs) given respectively by $f_j(.)$ and $F_j(.)$, $j = 1, 2$. Let, k and m are prior integers and ideal test time τ are determined, then, the ordered lifetime sample $(T_1,\ T_2,\ ...,\ T_D)$ which is constructed from the sample $\{X_1, X_2, ..., X_{M_D}, Y_1, Y_2, ..., Y_{N_D}\}$ with $D = M_D + N_D$ and D is defined by Eq. (3) to be k, m or integer such that $k < D < m$ is called joint Type-I GHS sample. Hence, for each random lifetime in the joint Type-I GHSC is described with time and type $(T,\ \eta)$. Then, $\underline{T} = ((T_1, \eta_1), (T_2, \eta_2), ..., (T_D, \eta_D))$ with $1 \leq D \leq M+N$ and the value of η_i take the value (1 or 0) depends on X or Y failure. Let $D_1 = \sum_{i=1}^{D} \eta_i$ denoted to the number of units fails from the line Φ_1 and $D_2 = \sum_{i=1}^{D} (1-\eta_i)$ denoted to the number of units fails from the line Φ_2. Then, the joint likelihood function of the observed sample $\underline{t} = ((t_1, \eta_1), (t_2, \eta_2), ..., (t_D, \eta_D))$ is given by

$$L(\underline{t}) = \frac{M!N!}{(M-D_1)!(N-D_2)!} \left[\prod_{i=1}^{D} [f_1(t_i)]^{\eta_i} [f_2(t_i)]^{1-\eta_i} \right] [S_1(C)]^{M-D_1} [S_2(C)]^{N-D_2}, \tag{4}$$

where $S_j(.)$, $j = 1, 2$ denoted to reliability functions and C is t_k, t_m or t_D corresponding to the value of D given in Eq. (3).

Under considerations that, the PDFs of the experimental unit come from lines Φ_1 and Φ_2 is Weibull distributed with PDFs is given by

$$f_j(t) = \alpha_j \beta_j t^{\alpha_j - 1} \exp\left(-\beta_j t^{\alpha_j}\right), \; t > 0, \; \alpha_j, \; \beta_j > 0, \; j = 1, \; 2. \tag{5}$$

And CDFs, reliability functions $S_j(.)$, and hazard rate functions $H_j(.)$ of the Weibull distributions are given, respectively, by

$$F_j(t) = 1 - \exp\left(-\beta_j t^{\alpha_j}\right), \tag{6}$$

$$S_j(t) = \exp\left(-\beta_j t^{\alpha_j}\right), \tag{7}$$

and

$$H_j(t) = \alpha_j \beta_j t^{\alpha_j - 1}. \tag{8}$$

3. MAXIMUM LIKELIHOOD ESTIMATION

For the joint Type-I GHS data $\underline{T} = ((T_1, \eta_1), \; (T_2, \eta_2), \; ..., \; (T_D, \eta_D))$, the likelihood function (4) with Weibull lifetime distributions in Eqs. (5) and (6) is reduced to

$$\begin{aligned} L(\alpha_1, \beta_1, \alpha_2, \beta_2 | \underline{t}) \quad &\propto \quad (\alpha_1 \beta_1)^{D_1} (\alpha_2 \beta_2)^{D_2} \exp\left\{ (\alpha_1 - 1) \sum_{i=1}^{D} \eta_i \log t_i - \beta_1 \sum_{i=1}^{D} \eta_i t_i^{\alpha 1} \right. \\ &+ \quad (\alpha_2 - 1) \sum_{i=1}^{D} (1 - \eta_i) \log t_i - \beta_2 \sum_{i=1}^{D} (1 - \eta_i) t_i^{\alpha 2} \\ &- \quad \left. (M - D_1) \beta_1 C^{\alpha_1} - (N - D_2) \beta_2 C^{\alpha_2} \right\}. \end{aligned} \tag{9}$$

The likelihood function under the natural logarithm is reduced to

$$\begin{aligned} \ell(\alpha_1, \beta_1, \alpha_2, \beta_2 | \underline{t}) \quad &= \quad D_1 \log(\alpha_1 \beta_1) + D_2 \log(\alpha_2 \beta_2) + (\alpha_1 - 1) \sum_{i=1}^{D} \eta_i \log t_i \\ &- \quad \beta_1 \sum_{i=1}^{D} \eta_i t_i^{\alpha 1} + (\alpha_2 - 1) \sum_{i=1}^{D} (1 - \eta_i) \log t_i - \beta_2 \sum_{i=1}^{D} (1 - \eta_i) t_i^{\alpha 2} \\ &- \quad (M - D_1) \beta_1 C^{\alpha_1} - (N - D_2) \beta_2 C^{\alpha_2}. \end{aligned} \tag{10}$$

3.1 Point Estimation

MLE is a commonly used method for parameters estimation, more detail see [16-18]. The likelihood equations are obtained from Eq. (10) by equating the first partial derivatives respect to parameters vector $\Psi = (\alpha_1, \; \beta_1, \; \alpha_2, \; \beta_2)$ to zero, then

$$\frac{\partial \ell(\alpha_1, \beta_1, \alpha_2, \beta_2 | \underline{t})}{\partial \beta_j} = 0, \; j = 1, \; 2,$$

are reduced to

$$\beta_1 = \frac{D_1}{\sum_{i=1}^{D} \eta_i t_i^{\alpha 1} + (M - D_1)C^{\alpha_1}}, \tag{11}$$

and

$$\beta_2 = \frac{D_2}{\sum_{i=1}^{D} (1-\eta_i) t_i^{\alpha 2} + (N - D_2)C^{\alpha_2}}. \tag{12}$$

Also,

$$\frac{\partial \ell(\alpha_1, \beta_1, \alpha_2, \beta_2 | \underline{t})}{\partial \alpha_j} = 0, \ j = 1, \ 2,$$

are reduced to

$$\frac{D_1}{\alpha_1} + \sum_{i=1}^{D} \eta_i \log t_i - \beta_1 \sum_{i=1}^{D} \eta_i t_i^{\alpha 1} \log t_i - (M - D_1)\beta_1 C^{\alpha_1} \log C = 0, \tag{13}$$

and

$$\frac{D_2}{\alpha_2} + \sum_{i=1}^{D} (1-\eta_i) \log t_i - \beta_2 \sum_{i=1}^{D} (1-\eta_i) t_i^{\alpha 1} \log t_i - (N - D_2)\beta_2 C^{\alpha_2} \log C = 0. \tag{14}$$

Then, the likelihood equations are reduced to two nonlinear Eqs. (13) and (14) which solve with any iteration method such as Newton Raphson or fixed point to obtain $\hat{\alpha}_1$ and $\hat{\alpha}_2$ and hence, the maximum likelihood estimates of β_1 and β_2 are obtained by substituting in Eqs. (11) and (12).

Remark: Eqs. (11)-(12) showed that under consideration of $D_1 = 0$ then α_1 and β_1 do not exist. Also, $D_2 = 0$ then α_2 and β_2 do not exist. Also, The exact distributions for estimators $\hat{\Psi} = (\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)$ is difficult to obtain see [15].

3.2 Approximate Interval Estimation

The approximate confidence intervals for the model parameters $\Psi = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ under the large sample approximation can be obtain from approximate Fisher information matrix of the parameters $\Omega = -E\left(\frac{\partial^2 \ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial \Psi_i \partial \Psi_j}\right)$, $i, \ j = 1, 2, 3, 4$. In different cases the minus expectation of second partially derivative of log-likelihood function cant be obtain. Hence, we can replace it by the estimate $\Omega_0(\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)$. Then, the interval estimation of the parameters $\alpha_1, \beta_1, \alpha_2$ and β_2 can be presented by the asymptotic normality distribution of $\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2$ and $\hat{\beta}_2$ with mean $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ and variance covariance matrix $\Omega_0^{-1}(\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)$ as

$$(\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2) \to N\Big((\alpha_1, \beta_1, \alpha_2, \beta_2), \Omega_0^{-1}(\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)\Big), \tag{15}$$

where $\Omega_0(\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2)$ is considered as observed information matrix presented by

$$\Omega_0(\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2) = \begin{bmatrix} -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_1^2} & -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_1\partial\beta_1} \\ -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_1\partial\alpha_1} & -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_1^2} \\ -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|v)}{\partial\alpha_2\partial\alpha_1} & -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_2\partial\beta_1} \\ -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_2\partial\alpha_1} & -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|v)}{\partial\beta_2\partial\beta_1} \end{bmatrix} \begin{matrix} -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_1\partial\alpha_2} & -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_1\partial\beta_2} \\ -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_1\partial\alpha_2} & -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_1\partial\beta_2} \\ -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_2^2} & -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_2\partial\beta_2} \\ -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_2\partial\alpha_2} & -\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_2^2} \end{matrix} \Bigg]_{\text{at } (\hat{\alpha}_1,\hat{\beta}_1,\hat{\alpha}_2,\hat{\beta}_2)} \tag{16}$$

Where

$$\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_j^2} = \frac{-D_j}{\beta_j^2},\ j=1,2, \tag{17}$$

$$\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\underline{t}|\underline{t})}{\partial\alpha_1\partial\beta_1} = \frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_1\partial\alpha_1} = -\sum_{i=1}^{D}\eta_i t_i^{\alpha_1}\log t_i - (M-D_1)C^{\alpha_1}\log C, \tag{18}$$

$$\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_1\partial\alpha_2} = \frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_2\partial\alpha_1} = 0, \tag{19}$$

$$\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_1\partial\beta_2} = \frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_2\partial\alpha_1} = 0, \tag{20}$$

$$\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_1^2} = \frac{-D_1}{\alpha_1^2} - \beta_1\sum_{i=1}^{D}\eta_i t_i^{\alpha 1}(\log t_i)^2 - (M-D_1)\beta_1 C^{\alpha_1}(\log C)^2, \tag{21}$$

$$\frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\alpha_2^2} = \frac{-D_2}{\alpha_2^2} - \beta_2\sum_{i=1}^{D}(1-\eta_i)t_i^{\alpha 2}(\log t_i)^2 - (N-D_2)\beta_2 C^{\alpha_2}(\log C)^2, \tag{22}$$

and

$$\begin{aligned} \frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\underline{t}|\underline{t})}{\partial\alpha_2\partial\beta_2} &= \frac{\partial^2\ell(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\partial\beta_2\partial\alpha_1} \\ &= -\sum_{i=1}^{D}(1-\eta_i)t_i^{\alpha 2}\log t_i - (N-D_2)C^{\alpha_2}\log C. \end{aligned} \tag{23}$$

Then, the 100(1–2γ)% approximate confidence intervals for α_1, β_1, α_2 and β_2 respectively given by

$$\begin{cases} \hat{\alpha}_1 \mp z_\gamma\sqrt{q_{11}} \\ \hat{\beta}_1 \mp z_\gamma\sqrt{q_{22}} \\ \hat{\alpha}_2 \mp z_\gamma\sqrt{q_{33}} \\ \hat{\beta}_2 \mp z_\gamma\sqrt{q_{44}} \end{cases}, \tag{24}$$

where the diagonal of the covariance matrix Ω_0^{-1} present the values q_{11}, q_{22}, q_{33} and q_{44} and the value z_γ is the percentile of the normal (0,1) with right-tail probability γ.

4. BAYESIAN MCMC ESTIMATION

In this section, we discuss Bayes estimators for the unknown parameters as well as the corresponding credible intervals under joint Type-I GHCS. This problem needs some assumptions about the form of the prior distributions for the unknown model parameters $\Psi = (\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2)$, the informative gamma prior densities are considered for each parameters as follows

$$\pi_i^*(\Psi_i) \propto \Psi_i^{a_i-1}\exp(-b_i\Psi i),\ \Psi_i > 0,\ (a_i,\ b_i > 0),\ i = 1,2,3,4, \tag{25}$$

where $\Psi_1 = \alpha_1$, $\Psi_2 = \beta_1$, $\Psi_3 = \alpha_2$, and $\Psi_4 = \beta_2$. Hence, the joint prior density presented by

$$\pi^*(\alpha_1,\beta_1,\alpha_2,\beta_2) \propto \prod_{i=1}^{4}\Psi_i^{a_i-1}\exp(-b_i\Psi_i). \tag{26}$$

From the likelihood function (9) and prior density (26) the joint posterior density function $\pi(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})$ can be built by

$$\pi(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t}) = \frac{\pi^*(\alpha_1,\beta_1,\alpha_2,\beta_2)L(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}{\int_\Psi \pi^*(\alpha_1,\beta_1,\alpha_2,\beta_2)L(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})d\alpha_1 d\beta_1 d\alpha_2 d\beta_2}. \tag{27}$$

Also the Byes estimtors for any function of the parameters $g(\alpha_1,\beta_1,\alpha_2,\beta_2)$ under squared error loss function (SEL) is given by

$$\begin{aligned} \hat{g}_B &= E_{\pi(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})}(g(\alpha_1,\beta_1,\alpha_2,\beta_2)) \\ &= \int_\Psi g(\alpha_1,\beta_1,\alpha_2,\beta_2)\pi(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t})d\alpha_1 d\beta_1 d\alpha_2 d\beta_2. \end{aligned} \tag{28}$$

Eq. (28) has a ratio of two integral which can be approximate with different methods such as numerical integration and Lindely approximation. One of the most important methods which can be applied is MCMC method describe as follows.

MCMC Approach

Since, the variety types of MCMC schemes, the formulation of posterior distribution determine the type of MCMC schemes which is applied. From the different available schemes of MCMC method, the important sub-class of them is Gibbs algorithms or in general Metropolis Hasting (MH) under Gibbs. When compare MCMC method with MLEs, it has advantage of obtaining a reasonable interval estimate of the unknown model parameters from empirical posterior distribution. This property is also true of any real function of the model parameters

The joint posterior density function of α_1, β_1, α_2, and β_2 can be written as

$$\begin{aligned}\pi(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{t}) &\propto \alpha_1^{a_1+D_1-1}\beta_1^{a_2+D_1-1}\alpha_2^{a_3+D_2-1}\beta_2^{a_4+D_2-1}\exp\{-b_1\alpha_1-b_2\beta_1\\ &- b_3\alpha_2-b_4\beta_2(\alpha_1-1)\sum_{i=1}^{D}\eta_i\log t_i-\beta_1\sum_{i=1}^{D}\eta_i t_i^{\alpha 1}\\ &+ (\alpha_2-1)\sum_{i=1}^{D}(1-\eta_i)\log t_i-\beta_2\sum_{i=1}^{D}(1-\eta_i)t_i^{\alpha 2}\\ &- (M-D_1)\beta_1 C^{\alpha_1}-(N-D_2)\beta_2 C^{\alpha_2}\}. \end{aligned} \tag{29}$$

From the joint posterior distribution in Eq. (29), the conditional posterior PDF's of model parameters are defined as follows

$$\beta_1|(\alpha_1,\alpha_2,\beta_2,\underline{t}) \to \text{Gamma}(a_2+D_1,U_1), \tag{30}$$

$$\beta_2|(\alpha_1,\alpha_2,\beta_1,\underline{t}) \to \text{Gamma}(a_4+D_2,U_2), \tag{31}$$

where

$$U_1 = b_2+\sum_{i=1}^{D}\eta_i t_i^{\alpha 1}+(M-D_1)C^{\alpha_1}, \tag{32}$$

and

$$U_2 = b_4+\sum_{i=1}^{D}(1-\eta_i)t_i^{\alpha_2}+(N-D_2)C^{\alpha_1}, \tag{33}$$

$$\begin{aligned}\alpha_1|(\beta_1,\alpha_2,\beta_2,\underline{t}) &\propto \alpha_1^{a_1+D_1-1}\exp\left\{-b_1\alpha_1+\alpha_1\sum_{i=1}^{D}\eta_i\log t_i\right.\\ &- \left.\beta_1\sum_{i=1}^{D}\eta_i t_i^{\alpha 1}-(M-D_1)\beta_1 C^{\alpha_1}\right\}, \end{aligned} \tag{34}$$

and

$$\begin{aligned}\alpha_2|(\alpha_1,\beta_1,\beta_2,\underline{t}) &\propto \alpha_2^{a_3+D_2-1}\exp\left\{-b_3\alpha_2+\alpha_2\sum_{i=1}^{D}(1-\eta_i)\log t_i\right.\\ &- \left.\beta_2\sum_{i=1}^{D}(1-\eta_i)t_i^{\alpha 2}-(N-D_2)\beta_2 C^{\alpha_2}\right\}. \end{aligned} \tag{35}$$

The two conditional distribution of parameters α_1 and α_2 given by Eqs. (34) and (35) are more similar to normal populations. Then, the operation of generate data from these distributions are built with MH algorithms see Metropolis *et al*. [19] under normal proposal distributions as follows.

MCMC algorithms (MH under Gibbs sampling)

Step 1: Put the initial vector $\Psi^{(0)} = (\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)$ and the indicator $\rho = 1$.

Step 2: From equations (30) and (31) two values $\beta_1^{(\rho)}$ and $\beta_2^{(\rho)}$ are generated from conditional gamma densities.

Step 3: Under normal proposal distributions of two values $\alpha_1^{(\rho)}$ and $\alpha_2^{(\rho)}$ are generated with MH algorithms.

Step 4: Then, the vector $\Psi^{(\rho)} = (\alpha_1^{(\rho)}, \beta_1^{(\rho)}, \alpha_2^{(\rho)}, \beta_2^{(\rho)})$ is constructed.

Step 5: Put $\rho = \rho + 1$.

Step 6: Steps from $2-5$ are repeted S times.

Step 7: If S^* is the MCMC number that is needing to achieved the stationary distribution (burn-in), then the Bayes MCMC point estimate of Ψ is given by

$$\hat{\Psi}_B = E(\Psi|\underline{t}) = \frac{1}{S-S^*} \sum_{i=S^*+1}^{S} \Psi^{(i)}, \tag{36}$$

and the corresponding posterior variance of Ψ is given by

$$\widehat{V}(\Psi|\underline{t}) = \frac{1}{S-S^*} \sum_{i=S^*+1}^{S} \left(\Psi^{(i)} - \hat{\Psi}_B\right)^2. \tag{37}$$

Step 8: After arrang the vector Ψ in aseding order, the corsponding $100(1-2\gamma)\%$ credible interval of Ψ is given by

$$\left(\Psi_{\gamma(S-S^*)}, \varphi_{(1-\gamma)(S-S^*)}\right), \tag{38}$$

where $\Psi = (\ \alpha_1, \beta_1, \alpha_2, \beta_2)$.

5. ILLUSTRATIVE EXAMPLE

Different threoritical results devolped in this artical are discussed through this section with a simulated data set as follows. Under given the prior parmeters a_i, b_i, $i = 1, 2, 3, 4$ generate a sample of size 100 and the true parameter is selected to be the mean of this samle. Hence, for the given ($a_1 = 5$, $b_1 = 3$) and ($a_2 = 1$, $b_2 = 3$) the true parameters values are selected to be $\alpha_1 = 1.73$ and $\beta_1 = 0.39$. Also, for given ($a_3 = 5$, $b_3 = 2$), and ($a_4 = 4$, $b_4 = 4$) the true parameters values are selected to be $\alpha_2 = 2.7$ and $\beta_2 = 0.7$. Then with the parameter vector $\Psi = (1.73, 0.39, 2.7, 0.7)$ and given $M = N = 30$, $(k, m) = (20, 30)$ and $\tau = 1.0$

From Weibull distribution with parameters $(\alpha_1, \beta_1) = (1.73, 0.39)$ generate a sample of size $M = 30$ as follows (data from line Φ_1)

$\underline{X}$={0.4974, 0.5250, 0.5741, 0.6362, 0.6377, 0.7313, 0.7571, 0.7919, 0.8199, 0.9143, 1.1239, 1.1522, 1.2628, 1.3187, 1.3874, 1.4449, 1.4733, 1.5024, 1.5264, 1.5781, 1.6429, 1.8642, 1.9706, 2.1925, 2.3227, 2.4023, 2.4113, 2.4515, 2.8872, 2.9222}.

Also, from Weibull distribution with parameters $(\alpha_2, \beta_2) = (2.7, 0.7)$ generate a sample of size $M = 30$ as follows (data from line Φ_2)

$\underline{Y}$={0.2585, 0.3766, 0.6252, 0.8052, 0.8102, 0.8430, 0.8543, 0.8586, 0.8758, 0.8909, 0.8942, 0.9236, 0.9687, 0.9962, 1.0057, 1.0332, 1.0540, 1.1154, 1.1213, 1.1668, 1.1697, 1.2849, 1.2971, 1.3847, 1.4399, 1.5787, 1.6376, 1.6382, 1.6981, 1.8230}.

Table 1. The joint Type-I GHS data with (k, m) = (20, 30) and $\tau = 1$.

$d = 24$									
0.2585	0.3766	0.4974	0.5250	0.5741	0.6252	0.6363	0.6377	0.7313	0.7571
0	0	1	1	1	0	1	1	1	1
0.7919	0.8052	0.8102	0.8199	0.8430	0.8543	0.8586	0.8758	0.8909	0.8942
1	0	0	1	0	0	0	0	0	0
0.9143	0.9236	0.9687	0.9962						
1	0	0	0						

Table 2. The point and 95% confidence intervals (ACIs and CIs) of MLEs Bayes estimates.

Pa.s	$(.)_{ML}$	$(.)_{BMCMC}$	95% ACIs	Length	95% CIs	Length
α_1 =1.73	2.8985	2.3629	(1.2118, 4.5851)	3.3732	(1.3842, 3.5626)	2.1785
β_1 =0.39	0.4216	0.4012	(0.1582, 0.6849)	0.5267	(0.1982, 0.6765)	0.4783
α_2 =2.70	3.9742	4.6326	(1.9875, 5.9609)	3.9734	(1.2056, 9.5594)	8.3538
β_2 =0.70	0.6065	0.6718	(0.2857, 0.9273)	0.6416	(0.3939, 1.0412)	0.6473

Then from two samples with value $(k, m) = (20, 30)$ and $\tau = 1.0$, the observed joint Type-I GHC data given in Table 1. From the data given in Table 1 the point MLE and Bayes MCMC estimate are given in Table 2. Also, the corresponding 95% approximate and credible intervals are given in Table 2. Figs. 2-5 show simulation number of the model parameters generated by MCMC method and the corresponding histogram. This figures show that the convergence in generation data from the posterior distribution under MCMC algorithms.

6. SIMULATION STUDIES

The theoretical results of two ML and Bayes estimates developed in this article are compared and assessed by building Monte Carlo simulation studies. In this problem, we measure the effect of change sample sizes (M, N), affect sample size and time (k, m, τ) and parameters values. The two terms average (AVG) and mean square error (MSE) are used to measure the validity of the point estimates as follows

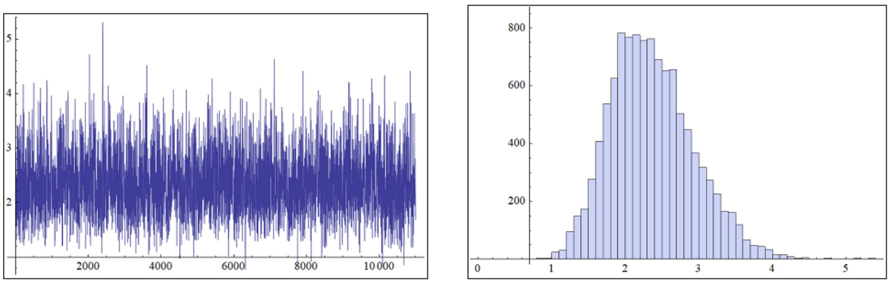

Fig. 2. Simulation number of α_1 and the corresponding histogram generated by MCMC method.

Table 3. The AVGs and MSEs of estimates with $(\alpha_1, \beta_1, \alpha_2, \beta_2) = (2.0, 0.3, 3.0, 0.5)$.

(M,N)	(k,m,τ)	Pa.	ML		$\text{BMCMC}_{\text{prior}_0}$		$\text{BMCMC}_{\text{prior}_1}$	
			AVGs	MSEs	AVGs	MSEs	AVGs	MSEs
(30, 30)	(20, 30, 1.0)	α_1	2.2045	0.4607	2.2029	0.4625	2.1949	0.4007
		β_1	0.3330	0.0510	0.3311	0.0502	0.3372	0.0444
		α_2	3.3062	0.9151	3.3178	0.9124	3.2464	0.7261
		β_2	0.5331	0.0812	0.5312	0.0801	0.5771	0.0744
(30, 30)	(30, 50, 1.0)	α_1	2.2039	0.3633	2.1002	0.3600	2.1789	0.3432
		β_1	0.3311	0.0422	0.3397	0.0402	0.3256	0.0351
		α_2	3.1268	0.7892	3.1285	0.7872	3.1005	0.5812
		β_2	0.5221	0.0582	0.5201	0.0573	0.5251	0.0493
(30, 30)	(20, 30, 1.5)	α_1	2.1974	0.3875	2.1944	0.3876	2.1973	0.3669
		β_1	0.3365	0.0667	0.3344	0.0657	0.3705	0.0361
		α_2	3.2872	0.8274	3.2852	0.8476	3.2908	0.6215
		β_2	0.5321	0.0589	0.5210	0.0622	0.5261	0.0553
(30, 30)	(30, 50, 1.5)	α_1	2.142	0.3532	2.1212	0.3501	2.1222	0.3234
		β_1	0.3213	0.0418	0.3295	0.0404	0.3201	0.0320
		α_2	3.1281	0.7512	3.1244	0.7570	3.1135	0.5412
		β_2	0.5157	0.0552	0.5231	0.0553	0.5209	0.0413
(50, 50)	(40, 60, 1.0)	α_1	2.1354	0.4341	2.1472	0.4312	2.1777	0.3707
		β_1	0.3231	0.0492	0.3241	0.0482	0.3241	0.0412
		α_2	3.2145	0.8053	3.2174	0.8100	3.2400	0.6560
		β_2	0.5232	0.0752	0.5214	0.0743	0.5222	0.0581
(50, 50)	(50, 70, 1.0)	α_1	2.1221	0.3213	2.1404	0.3337	2.1421	0.2841
		β_1	0.3124	0.0601	0.3114	0.0597	0.3095	0.0251
		α_2	3.2130	0.707	3.1882	0.6976	3.1158	0.5115
		β_2	0.5121	0.0519	0.5109	0.0502	0.5111	0.0453
(50, 50)	(40, 60, 1.5)	α_1	2.1884	0.3772	2.1774	0.3772	2.1883	0.3462
		β_1	0.3361	0.0655	0.3320	0.0614	0.3700	0.0354
		α_2	3.2869	0.8271	3.2844	0.8466	3.2888	0.6209
		β_2	0.5312	0.0578	0.5203	0.0608	0.5242	0.0511
(50, 50)	(50, 70, 1.5)	α_1	2.1051	0.3011	2.1312	0.3124	2.1055	0.2229
		β_1	0.3039	0.0582	0.3099	0.0527	0.3017	0.0223
		α_2	3.2110	0.7040	3.1771	0.6612	3.1113	0.5007
		β_2	0.5022	0.0491	0.50188	0.0422	0.5221	0.0402

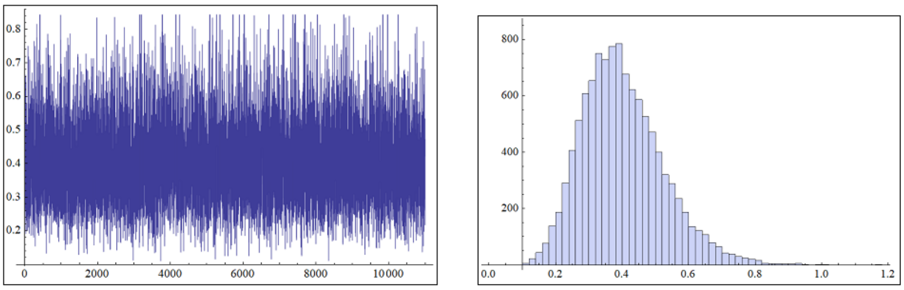

Fig. 3. Simulation number of β_1 and the corresponding histogram generated by MCMC method.

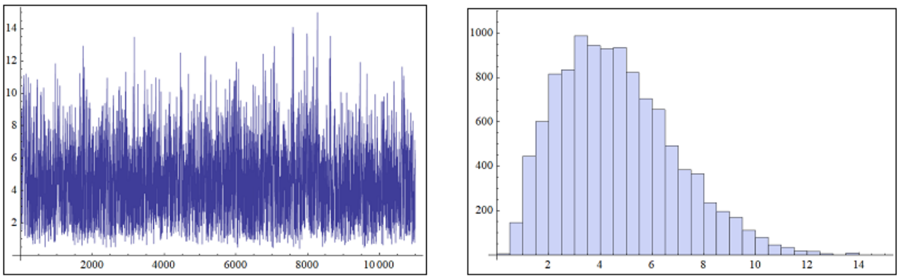

Fig. 4. Simulation number of α_2 and the corresponding histogram generated by MCMC method.

$$\text{AVG} = \overline{\Psi} = \frac{1}{\kappa}\sum_{i=1}^{\kappa}\hat{\Psi}^{(i)} \text{ and MSE=}\frac{1}{\kappa}\sum_{i=1}^{\kappa}\left(\hat{\Psi}^{(i)} - \overline{\Psi}\right)^2 \tag{39}$$

where $\Psi = (\ \alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2)$ denoted to populations parameters. Also, two terms average interval length (AL) and probability coverage (PC) are used to measure the validity of the each approximate confidence intervals and credible intervals. Hence, two sets of populations parameters are selected $(\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2) = \{(2.0,\ 0.3,\ 3.0,\ 0.5),\ (0.6,\ 1.0,\ 0.8,\ 1.2)\}$. The prior parameters are selected to be $E(\Psi_i) \simeq \frac{a_i}{b_i}$, where $(\Psi_1 = \alpha_1,\ \Psi_2 = \beta_1,\ \Psi_3 = \alpha_2,\ \Psi_4 = \beta_2)$ .For the prior information, we consider two cases, the first case in which the joint posterior distribution is proportional with likelihood function, called non-informative priors, priors$_0$. The second case is informative prior information, we consider prior$_1$: $(a_1 = 4,\ a_2 = 3,\ a_3 = 3,\ a_4 = 2,\ b_1 = 2,\ b_2 = 5,\ b_3 = 1,\ b_4 = 1)$ for the first set of parameters, prior$_2$: $(a_1 = 1.5,\ a_2 = 3.0,\ a_3 = 2,\ a_4 = 2.0,\ b_1 = 3.0,\ b_2 = 3.0\ ,\ b_3 = 2.0,\ b_4 = 2.5)$ for the second set of parameters. The Bayes estimate considered under squared error loss function, also the Bayes point and interval estimates computed with 11000 iteration of MCMC with 1000 is considered as burn-in. The simulation process is constructed with 1000 times and the corresponding AG, MES, AL and PC values of estimates are computed in results are reported in Tables 3-6.

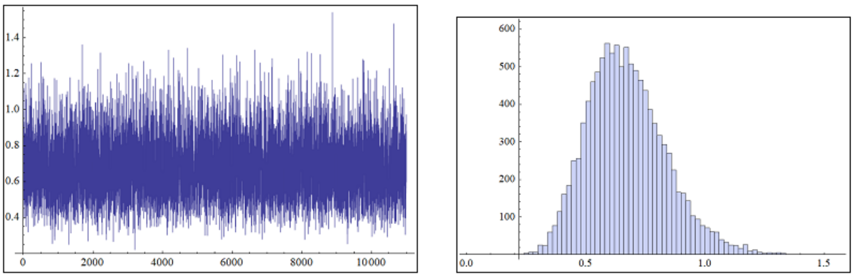

Fig. 5. Simulation number of β_2 and the corresponding histogram generated by MCMC method.

Table 4. The CPs and ALs for the interval estimates with $(\alpha_1, \beta_1, \alpha_2, \beta_2) = (2.0, 0.3, 3.0, 0.5)$.

(M,N)	(k,m,τ)	Pa.	ML		$BMCMC_{0:prior}$		$BMCMC_{1:prior}$	
			PCs	ALs	PCs	ALs	PCs	ALs
(30, 30)	(20, 30, 1.0)	α_1	0.91	3.2722	0.90	3.2710	0.92	3.1142
		β_1	0.90	1.4520	0.90	1.4534	0.92	1.2255
		α_2	0.92	5.6522	0.92	5.4448	0.92	4.1282
		β_2	0.92	2.3922	0.96	2.3892	0.96	2.1477
(30, 30)	(30, 50, 1.0)	α_1	0.93	3.1472	0.92	3.1120	0.93	3.0084
		β_1	0.92	1.4109	0.93	1.4213	0.96	1.2001
		α_2	0.91	5.6231	0.91	5.4217	0.93	4.1002
		β_2	0.93	2.3832	0.94	2.3621	0.92	2.1274
(30, 30)	(20, 30, 1.5)	α_1	0.91	3.2701	0.91	3.2699	0.93	3.1133
		β_1	0.92	1.4503	0.93	1.4517	0.92	1.2240
		α_2	0.92	5.6501	0.92	5.4432	0.94	4.1269
		β_2	0.91	2.3900	0.92	2.3885	0.91	2.1466
(30, 30)	(30, 50, 1.5)	α_1	0.94	3.1444	0.95	3.1120	0.94	3.0012
		β_1	0.92	1.4089	0.93	1.4188	0.96	1.1985
		α_2	0.92	5.6201	0.92	5.4175	0.95	4.0894
		β_2	0.94	2.3807	0.94	2.3512	0.93	2.1150
(50, 50)	(40, 60, 1.0)	α_1	0.93	3.1212	0.94	3.1004	0.94	2.8512
		β_1	0.93	1.4012	0.93	1.4004	0.95	1.1650
		α_2	0.93	5.6120	0.93	5.4122	0.95	4.0610
		β_2	0.94	2.3611	0.92	2.3411	0.92	2.1066
(50, 50)	(50, 70, 1.0)	α_1	0.92	3.1001	0.93	3.0821	0.95	2.8320
		β_1	0.93	1.3964	0.93	1.3754	0.95	1.1410
		α_2	0.94	5.6002	0.94	5.4001	0.94	4.0390
		β_2	0.94	2.3312	0.93	2.3098	0.93	2.0874
(50, 50)	(40, 60, 1.5)	α_1	0.93	3.1202	0.94	3.0952	0.94	2.8500
		β_1	0.92	1.4001	0.96	1.3952	0.94	1.1638
		α_2	0.93	5.6107	0.93	5.4101	0.95	4.0590
		β_2	0.95	2.3591	0.94	2.3399	0.93	2.1042
(50, 50)	(50, 70, 1.5)	α_1	0.92	2.875	0.93	3.0741	0.96	2.8115
		β_1	0.92	1.3900	0.94	1.3702	0.95	1.1350
		α_2	0.94	5.5963	0.94	5.329	0.94	4.0352
		β_2	0.95	2.3225	0.94	2.3045	0.92	2.0819

Table 5. The AVGs and MSEs of estimates with $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ at (0.6, 1.0, 0.8, 1.2).

(M,N)	(k,m,τ)	Pa.	ML		$\text{BMCMC}_{\text{prior}_0}$		$\text{BMCMC}_{\text{prior}_2}$	
			AVGs	MSEs	AVGs	MSEs	AVGs	MSEs
(30, 30)	(20, 30, 0.5)	α_1	0.8541	0.2473	0.8334	0.2128	0.8017	0.1352
		β_1	1.3540	0.5421	1.3112	0.5289	1.2471	0.4165
		α_2	1.0042	0.5437	0.9892	0.5178	0.9088	0.3215
		β_2	1.4242	0.5847	1.4124	0.5669	1.3124	0.4665
(30, 30)	(30, 50, 0.5)	α_1	0.8312	0.1245	0.8289	0.1154	0.8201	0.0998
		β_1	1.2345	0.2143	1.2118	0.2054	1.2000	0.1009
		α_2	0.9872	0.1542	0.9749	0.1507	0.8521	0.0984
		β_2	1.3985	0.2415	1.3777	0.2311	1.3421	0.1328
(30, 30)	(20, 30, 1.3)	α_1	0.8332	0.2408	0.8278	0.2099	0.8118	0.1307
		β_1	1.3475	0.5364	1.3077	0.5203	1.2321	0.4081
		α_2	09872	0.5345	0.9799	0.5103	0.8562	0.3041
		β_2	1.4211	0.5745	1.4090	0.5559	1.3021	0.4598
(30, 30)	(30, 50, 1.3)	α_1	0.8285	0.1188	0.8145	0.1100	0.8197	0.0908
		β_1	1.2302	0.2078	1.2095	0.2004	1.1745	0.0999
		α_2	0.9801	0.1399	0.9701	0.1498	0.8489	0.0900
		β_2	1.3785	0.2332	1.3705	0.2217	1.3111	0.1231
(50, 50)	(40, 60, 0.5)	α_1	0.8154	0.1099	0.8103	0.1024	0.8104	0.0889
		β_1	1.2231	0.1987	1.2124	0.1990	1.1321	0.0910
		α_2	0.9321	0.1012	0.9001	0.1008	0.8401	0.0897
		β_2	1.3124	0.2012	1.3231	0.2008	1.2410	0.1124
(50, 50)	(50, 70, 0.5)	α_1	0.7542	0.0954	0.7401	0.0934	0.7123	0.0742
		β_1	1.2119	0.1231	1.2001	0.1124	1.1002	0.0864
		α_2	0.8632	0.0997	0.8547	0.0994	0.8307	0.0795
		β_2	1.274	0.1872	1.3112	0.1822	1.2245	0.1002
(50, 50)	(40, 60, 1.3)	α_1	0.8001	0.1014	0.7992	0.0999	0.7404	0.0812
		β_1	1.2124	0.1754	1.2004	0.1840	1.1119	0.0890
		α_2	0.9124	0.0989	0.9012	0.0997	0.8320	0.0874
		β_2	1.3078	0.1872	1.3090	0.1784	1.2210	0.1088
(50, 50)	(50, 70, 1.3)	α_1	0.7274	0.0872	0.7211	0.0824	0.6821	0.0700
		β_1	1.1745	0.1019	1.1721	0.1002	1.1121	0.0810
		α_2	0.8452	0.0875	0.8385	0.0861	0.8185	0.0707
		β_2	1.211	0.1521	1.2012	0.1487	1.2009	0.0997

7. CONCLUDING REMARKS

The problem of determine the relative merits of products in the competing duration with different lines of production has occupy important position in the last view years. In this section, we discussed this problem under ML and Bayesian estimations, for the unknown model parameters of two Weibull lifetime distributions under joint Type-I GHCS. Numerical results was conducted to assess and compare the performance our proposed methods. Then from this results we can see the following.

Table 6. The CPs and ALs for the interval estimates with $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ at (0.6, 1.0, 0.8, 1.2).

(M,N)	(k,m,τ)	Pa.	ML		$\text{BMCMC}_{\text{prior}_0}$		$\text{BMCMC}_{\text{prior}_2}$	
			PCs	ALs	PCs	ALs	PCs	ALs
(30, 30)	(20, 30, 0.5)	α_1	0.90	2.5214	0.91	2.4124	0.96	2.2147
		β_1	0.92	4.6527	0.92	4.5784	0.92	4.1245
		α_2	0.91	3.2157	0.92	3.2008	0.92	3.0189
		β_2	0.92	5.2364	0.90	5.2108	0.91	5.0024
(30, 30)	(30, 50, 0.5)	α_1	0.92	2.254	0.93	2.2104	0.94	2.0587
		β_1	0.93	4.4122	0.92	4.3201	0.92	3.9850
		α_2	0.91	3.0017	0.94	3.0174	0.94	2.8752
		β_2	0.93	5.0241	0.90	5.0001	0.93	4.7854
(30, 30)	(20, 30, 1.3)	α_1	0.93	2.5019	0.91	2.4002	0.93	2.2011
		β_1	0.93	4.6325	0.93	4.5524	0.95	4.1009
		α_2	0.96	3.2008	0.96	3.1897	0.94	2.9981
		β_2	0.94	5.2128	0.93	5.1842	0.94	4.8974
(30, 30)	(30, 50, 1.3)	α_1	0.93	2.2219	0.94	2.1874	0.95	2.0241
		β_1	0.93	4.3894	0.92	4.2985	0.92	3.9547
		α_2	0.96	2.8990	0.92	2.8892	0.95	2.8425
		β_2	0.93	4.8752	0.90	4.8521	0.95	4.7426
(50, 50)	(40, 60, 0.5)	α_1	0.94	2.2110	0.93	2.1624	0.96	2.0102
		β_1	0.92	4.3624	0.93	4.2745	0.93	3.9324
		α_2	0.93	2.8741	0.95	2.8632	0.95	2.8245
		β_2	0.91	4.8533	0.93	4.8324	0.93	4.7221
(50, 50)	(50, 70, 0.5)	α_1	0.95	2.0478	0.94	2.0004	0.95	1.8922
		β_1	0.91	4.124	0.96	4.0175	0.93	3.7451
		α_2	0.96	2.6124	0.93	2.6542	0.94	2.4210
		β_2	0.94	4.6523	0.93	4.6415	0.94	4.3217
(50, 50)	(40, 60, 1.3)	α_1	0.94	2.2032	0.93	2.1421	0.97	2.0001
		β_1	0.95	4.3421	0.93	4.2524	0.93	3.9123
		α_2	0.93	2.8524	0.95	2.8478	0.95	2.8001
		β_2	0.91	4.8336	0.94	4.8300	0.96	4.7099
(50, 50)	(50, 70, 1.3)	α_1	0.94	2.0233	0.95	1.9904	0.93	1.8524
		β_1	0.94	4.1009	0.96	4.1007	0.93	3.7218
		α_2	0.96	2.6003	0.95	2.6326	0.95	2.4013
		β_2	0.95	4.6234	0.93	4.6207	0.94	4.3101

1. Tables 3-6 show that, using the joint Type-I GHCS for lifetime Weibull products are more acceptable.
2. For two methods of estimation, Bayes method perform better than ML method.
3. The results of MLE are closed to one Bayes estimates under non-informative prior.
4. At the effective sample size (k, m) are increases, results of the MSEs and interval length are reduce.
5. The results perform better for the large value of test time τ.
6. Results of simulation study is more better for two cases of the parameters values.

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Ali Algarni received the Ph.D. degree in Statistics from the School of Engineering and Natural Sciences at the University of Wollogong, Australia. He is currently an Associate Professor of Applied Statistics at the Department of Statistics at King Abdul Aziz University (KAU), Saudi Arabia. Moreover, since 2016, he has been a General Supervisor of the Strategic Planning Unit at KAU and Consultant of the KAU's Vice President for Educational Affairs. His research interests include sampling theory, data mining, medical statistics, machine learning and big data analysis.

Abdullah M. Almarashi received the Ph.D. degree in Statistical Sciences (Time Series Analysis) from the School of Science, University of Strathclyde Glasgow, Uk in 2014. He is currently an Associate Professor with the Faculty of Science, Statistics Department, King Abdul Aziz University (KAU), and Saudi Arabia since 2019. Moreover, and since 2019, he has been a Consultant of the Vice Presidency of Educational Affairs for Strategic Planning, King Abdulaziz University. His research interests include time series analysis, statistical inference, medical statistics, and regression models.

G. A. Abd-Elmougod received the Ph.D. degree in Mathematical Statistics in 2012 from Sohag University. His research interests include different directions in statistics. He is the author of several articles published in different international scientific journals and is a member of different working groups. He is presently employed as an Assistant Professor of Mathematics Department, Faculty of Science, Damanhour University, Egypt.