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Steiner Distance in Join, Corona, Cluster, and Threshold Graphs

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For a connected graph G and a subset S of its vertices, the *Steiner tree* problem consists of finding a minimum-size connected subgraph containing S. The *Steiner distance* of S is the size of a Steiner tree for S, and the *Steiner k-diameter* of G is the maximum value of the Steiner distance over all vertex subsets S of cardinality k. Calculation of Steiner trees and Steiner distance is known to be NP-hard in general, so applications may benefit from using graphs where the Steiner distance and structure of Steiner trees are known. In this paper, we investigate the Steiner distance and Steiner k-diameter of the join, corona, and cluster of connected graphs, as well as threshold graphs.

Keywords: wireless sensor networks, localization, mobile beacon, mobile anchor, RSSI

1. INTRODUCTION

The Steiner tree problem in graphs was formulated in 1971 by Hakimi [1] and Levi [2]. In the case of an unweighted, undirected graph, this problem consists of finding, for a subset of vertices S, a minimum-size connected subgraph that contains the vertices in S. The size of the resulting subgraph is also known as the *Steiner distance* of S. The computational side of the Steiner tree problem has been widely studied, and it is known to be an NP-hard problem for general graphs [3]. The determination of a Steiner tree in a graph is a discrete analogue of the well-known geometric Steiner problem: In a Euclidean space (usually a Euclidean plane) find the shortest possible network of line segments connecting a set of given points.

Steiner trees have applications to multiprocessor computer networks, as well as other network structures. For example, it may be desired to connect a certain set of processors with a subnetwork that uses the least number of communication links. A Steiner tree for the vertices representing the processors that need to be connected corresponds to such a minimum subnetwork. Since computing such a Steiner tree is NP-hard in general, when designing networks for such applications it may be beneficial to simplify this computation by choosing networks where the size and structure of Steiner trees are

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known. In this paper, we consider the Steiner tree and Steiner distance problems, as well as the closely related *Steiner k-diameter* problem, for some particular families of graphs: namely, for the join and corona of connected graphs, as well as threshold graphs.

1.1 Preliminaries

All graphs in this paper are undirected, unweighted, finite and simple. We follow [4] in describing some frequently used notation and terminology, and refer the reader to [5] for notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$, $e(G)$, and $\delta(G)$ denote the set of vertices, the set of edges, the size, and the minimum degree, respectively. The degree of a vertex v in G is denoted by $deg_G(v)$. We denote by K_n, P_n, $K_{1,n-1}$ and C_n be the complete graph of order n, the path of order n, the star of order n, and the cycle of order n, respectively. For any subset X of $V(G)$, let $G[X]$ denote the subgraph of G induced by X; similarly, for any subset F of $E(G)$, let $G[F]$ denote the subgraph of G induced by F. We use $G\backslash X$ to denote the subgraph of G obtained by removing all the vertices of X together with the edges incident with them from G; similarly, we use $G\backslash F$ to denote the subgraph of G obtained by removing all the edges of F from G. If $X = \{v\}$ and $F = \{e\}$, we simply write $G\backslash v$ and $G\backslash e$ for $G\backslash\{v\}$ and $G\backslash\{e\}$, respectively. For two subsets X and Y of $V(G)$ we denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y. If $X = \{x\}$, we simply write $E_G[x, Y]$ for $E_G[\{x\}, Y]$.

1.2 Distance and its Generalizations

Distance is one of the most basic concepts of graph theory. If G is a connected graph and $u, v \in V(G)$, then the *distance* $d_G(u, v)$ between u and v is the length of a shortest path connecting u and v. If v is a vertex of a connected graph G, then the *eccentricity* $e(v)$ of v is defined by $e(v) = \max\{d_G(u, v) \mid u \in V(G)\}$. Furthermore, the *radius* $rad(G)$ and *diameter* $diam(G)$ of G are defined by $rad(G) = \min\{e(v) \mid v \in V(G)\}$ and $diam(G) = \max\{e(v) \mid v \in V(G)\}$. These last two concepts are related by the inequalities $rad(G) \leq diam(G) \leq 2rad(G)$.

The distance between two vertices u and v in a connected graph G also equals the minimum size of a connected subgraph of G containing both u and v. This observation suggests a generalization of distance. The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural generalization of the concept of classical graph distance. For a graph $G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T(V', E')$ of G that is a tree with $S \subseteq V'$. Let G be a connected graph of order at least 2 and let S be a nonempty set of vertices of G. Then the *Steiner distance* $d_G(S)$ among the vertices of S (or simply the distance of S) is the minimum size among all connected subgraphs whose vertex sets contain S. Note that if H is a connected subgraph of G such that $S \subseteq V(H)$ and $e(H) = d_G(S)$, then H is a tree. Observe that $d_G(S) = \min\{e(T) \mid S \subseteq V(T)\}$, where T is subtree of G. Furthermore, if $S = \{u, v\}$, then $d_G(S) = d_G(u, v)$ is the classical distance between u and v. Set $d_G(S) = \infty$ when there is no S-Steiner tree in G.

Observation 1.1: Let G be a graph of order n and k be an integer with $2 \leq k \leq n$. If $S \subseteq V(G)$ and $|S| = k$, then $d_G(S) \geq k - 1$.

Let n and k be two integers with $2 \leq k \leq n$. The *Steiner k-eccentricity* $e_k(v)$ of a vertex v of G is defined by $e_k(v) = \max\{d_G(S) | S \subseteq V(G), |S| = k, \text{ and } v \in S\}$. The *Steiner k-radius* of G is $srad_k(G) = \min\{e_k(v) | v \in V(G)\}$, while the *Steiner k-diameter* of G is $sdiam_k(G) = \max\{e_k(v) \mid v \in V(G)\}$. Note for every connected graph G that $e_2(v) = e(v)$ for all vertices v of G and that $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$.

Observation 1.2: Let k, n be two integers with $2 \leq k \leq n$.
(1) If H is a spanning subgraph of G, then $sdiam_k(G) \leq sdiam_k(H)$.
(2) For a connected graph G, $sdiam_k(G) \leq sdiam_{k+1}(G)$.

In [6], Chartrand, Okamoto, Zhang obtained the following upper and lower bounds of $sdiam_k(G)$.

Theorem 1.1: [6] Let k, n be two integers with $2 \leq k \leq n$, and let G be a connected graph of order n. Then $k - 1 \leq sdiam_k(G) \leq n - 1$. Moreover, the upper and lower bounds are sharp.

In [7], Dankelmann, Swart and Oellermann obtained a bound on $sdiam_k(G)$ for a graph G in terms of the order of G and the minimum degree δ of G, that is, $sdiam_k(G) \leq \frac{3n}{\delta+1} + 3k$. Later, Ali, Dankelmann, Mukwembi [12] improved the bound of $sdiam_k(G)$ and showed that $sdiam_k(G) \leq \frac{3n}{\delta+1} + 2k - 5$ for all connected graphs G. Moreover, they constructed graphs to show that the bounds are asymptotically best possible.

1.3 Product Networks and Threshold Graphs

Product networks were proposed based upon the idea of using the product as a tool for "combining" two known graphs with established properties to obtain a new one that inherits properties from both [8]. Familiar examples of operations used to produce product networks include the Cartesian product of graphs, the strong product, and the lexicographic product. The join, corona, and cluster operations can also be regarded as graph product operations [9, 10] for more details.

The join, corona, and cluster operations, as well as threshold graphs, are defined as follows.

The *join* or *complete product* of two disjoint graphs G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh | g \in V(G), h \in V(H)\}$.

The *corona* $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the ith copy of H with the ith vertex of G, where $i = 1, 2, \ldots, |V(G)|$.

The *cluster* [10] $G \odot H$ is obtained by taking one copy of G and $|V(G)|$ copies of a rooted graph H, and by identifying the root of the ith copy of H with the ith vertex of G, where $i = 1, 2, ..., |V(G)|$. The cluster can be viewed as a special case of the rooted product of graphs [11], where the family $H_1, \ldots, H_{|V(G)|}$ of rooted graphs to be joined to the base graph G consists of $|V(G)|$ copies of the same rooted graph.

A graph G is a *threshold graph*, if there exists a weight function w: $V(G) \to \mathbb{R}$ and a real constant t such that two vertices $g, g' \in V(G)$ are adjacent if and only if $w(g) + w(g') \geq t$.

In the following sections, we consider the Steiner distance and Steiner k-diameter problems for the join, corona, and cluster of connected graphs, as well as threshold graphs. The structure of the minimum Steiner trees can also be seen from the derivation of the Steiner distance. Each type of graph is treated in a separate section.

2. STEINER DIAMETER OF THE JOIN OF CONNECTED GRAPHS

We now give the exact value for Steiner distance of joined graphs.

Proposition 2.1: Let k, m, and n be three integers with $3 \leq k \leq m + n$, and let G and H be two connected graphs with n and m vertices, respectively. Let S be a set of distinct vertices of $G \vee H$ such that $|S| = k$.

(i) If $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$, then $d_{G\vee H}(S) = k - 1$.
(ii) If $S \cap V(H) = \emptyset$ and $G[S]$ is connected, then $d_{G\vee H}(S) = k - 1$; if $S \cap V(H) = \emptyset$ and $G[S]$ is not connected, then $d_{G\vee H}(S) = k$.
(iii) If $S \cap V(G) = \emptyset$ and $H[S]$ is connected, then $d_{G\vee H}(S) = k - 1$; if $S \cap V(G) = \emptyset$ and $H[S]$ is not connected, then $d_{G\vee H}(S) = k$.

***Proof*:** For (i), without loss of generality, let $S \cap V(G) = \{g_1, g_2, \ldots, g_x\}$ and $S \cap V(H) = \{h_1, h_2, \ldots, h_{k-x}\}$. Then the tree induced by the edges in $\{g_1h_1\} \cup \{g_1h_i \mid 2 \leq i \leq k - x\} \cup \{g_ih_1 \mid 2 \leq i \leq x\}$ is an S-Steiner tree, and so $d_{G\vee H}(S) \leq k - 1$. From Observation 1.1, we have $d_{G\vee H}(S) = k - 1$, as desired.

We only give the proof of (ii), and the proof of (iii) follows by symmetry of the join operation. Since $S \cap V(H) = \emptyset$, it follows that $S \subseteq V(G)$. If $G[S]$ is connected, there exists a spanning tree of $G[S]$, which is an S-Steiner tree in G. Therefore, we have $d_{G\vee H}(S) = d_G(S) \leq k - 1$, and hence $d_{G\vee H}(S) = k - 1$. If $G[S]$ is not connected, then $d_{G\vee H}(S) \geq k$. The tree induced by the edges in $E_{G\vee H}[h, S]$ is an S-Steiner tree, where $h \in V(H)$, and hence $d_{G\vee H}(S) \leq k$. So, we have $d_{G\vee H}(S) = k$. ❑

For Steiner diameter of joined graphs, we have the following.

Proposition 2.2: Let G be a connected graph with n vertices, and let H be a connected graph with $m (n \leq m)$ vertices. Let k be an integer with $3 \leq k \leq n + m$.

(i) If $k > m$, then $sdiam_k(G\vee H) = k - 1$.
(ii) If $n < k \leq m$ and $sdiam_k(H) = k - 1$, then $sdiam_k(G\vee H) = k - 1$; if $n < k \leq m$ and $sdiam_k(H) \geq k$, then $sdiam_k(G\vee H) = k$.
(iii) If $3 \leq k \leq n$, and $sdiam_k(G) \geq k$ or $sdiam_k(H) \geq k$, then $sdiam_k(G\vee H) = k$; If $3 \leq k \leq n$ and $sdiam_k(G) = sdiam_k(H) = k - 1$, then $sdiam_k(G\vee H) = k - 1$.

***Proof*:** We only give the proofs of (i) and (ii), and the proof of (iii) is similar and is omitted. For (i), since $k > m$, it follows that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$ for any $S \subseteq V$

$(G \vee H)$ and $|S| = k$. From (i) of Proposition 2.1, we have $d_{G\vee H}(S) = k - 1$. Therefore we have $sdiam_k(G\vee H) = k - 1$.

For (ii), for any $S \subseteq V(G\vee H)$ and $|S| = k$, we have $S \subseteq V(H)$, or $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$, since $n < k \leq m$. Suppose $sdiam_k(H) = k - 1$. If $S \subseteq V(H)$, then $d_{G\vee H}(S) = k - 1$, since $sdiam_k(H) = k - 1$. If $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$, then $d_{G\vee H}(S) = k - 1$ by (i) of Proposition 2.1. Therefore we have $sdiam_k(G\vee H) = k - 1$. Suppose $sdiam_k(H) \geq k$. Then we can see that $sdiam_k(G\vee H) \geq k$ by choosing $S \subseteq V(H)$ with $d_H(S) = k$. If $S \subseteq V(H)$, then the tree induced by the edges in $E_{G\vee H}[g, S]$ is an S-Steiner tree, where $g \in V(G)$, and hence $d_{G\vee H}(S) \leq k$. If $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$, then $d_{G\vee H}(S) = k - 1$ by (i) of Proposition 2.1. Therefore we have $sdiam_k(G\vee H) \leq k$, and we may conclude that $sdiam_k(G\vee H) = k$. ❑

3. STEINER DIAMETER OF THRESHOLD GRAPHS

To calculate the Steiner distance for threshold graphs, the following observations are useful.

Observation 3.1: Let $G(\{1, 2, ..., n\}, E)$ be a threshold graph with weight function w: $V(G) \to \mathbb{R}$ and threshold constant t. Let the vertices be labelled so that $w(1) \geq w(2) \geq \ldots \geq w(n)$. Then

(a) $d_1 \geq d_2 \geq \ldots \geq d_n$, where d_i is the degree of vertex i.
(b) $N(i) = \{1, 2, \ldots, d_i\}$ for every $i \in I$ with $d_i < i$, and $N(i) \cup \{i\} = \{1, 2, \ldots, d_i + 1\}$ when $d_i \geq i$. Furthermore, if G is connected, then every vertex in G is adjacent to 1.
(c) $I = \{i \in V(G)\colon d_i \leq i - 1\}$ is a maximum independent set of G and $G\backslash I$ is a clique in G.

***Proof*:** To see (a), note that if $w(i) \geq w(j)$, then $w(j) + w(k) \geq t$ implies that $w(i) + w(k) \geq t$, so $d_i \geq d_j$.

For (b), consider a fixed vertex i, and let j be the vertex with maximum label such that $w(i) + w(j) \geq t$. For any other vertex k, if $k > j$, then clearly i and k are not adjacent, and if $k \leq j$, then $w(i) + w(k) \geq w(i) + w(j) \geq t$, so i and k are adjacent. Thus if $j < i$ then $N(i) = \{1, 2, \ldots, j\}$ and $d_i = j$, and if $j \geq i$ then $N(i) \cup \{i\} = \{1, 2, \ldots, j\}$ and $d_i = j - 1$.

For (c), suppose that $i \in I$. Then for any $j \in V(G)$ with $j > i$, we also have $j \in I$, since $d_j \leq d_i \leq i - 1 < j - 1$. This also means that i and j are not adjacent by (b), since $d_i < j$. Thus no two members of I are adjacent, so I is an independent set of G. Now consider the set $G\backslash I = \{i \in V(G)\colon d_I > i - 1\}$. If $i \in G\backslash I$ and $j < i$, then we also have $j \in G\backslash I$, since $d_j \geq d_i > i - 1 > j - 1$, and we know that i and j are adjacent by (b) since $d_i \geq i > j$. Thus every two members of $G\backslash I$ are adjacent, so $G\backslash I$ is a clique in G.

Furthermore, we will now show that I is maximum. If a vertex of I is adjacent to every vertex of the clique $G\backslash I$, then I is clearly a maximum independent set, since only one of those $|G\backslash I| + 1$ vertices can be included in any independent set. Thus it suffices to show that I always contains a vertex adjacent to every vertex in $G\backslash I$. Let j be the vertex with minimum label in I. Then $(j - 1)$ is the element of $G\backslash I$ with maximum label, so $d_{j-1} > j - 2$ and thus $d_{j-1} \geq j - 1$. By (b), this implies that $\{1, 2, \ldots, j - 2, j\} \subseteq N(j - 1)$, so j and $(j - 1)$ are adjacent. Therefore $\{1, 2, \ldots, j - 1\} \subseteq N(j)$, and since$(j - 1)$ was the element of $G\backslash I$ with maximum label, j is adjacent to every vertex in $G\backslash I$. ❑

Let C_r and I_{n-r} denote the clique and the maximum independent set of G, respectively, with $V(C_r) = \{g_1, g_2, \ldots, g_r\}$ and $V(I_{n-r}) = \{g'_1, g'_2, \ldots, g'_{n-r}\}$ such that $d_G(g_1) \geq d_G(g_2) \geq \ldots \geq d_G(g_r)$ and $d_G(g'_1) \geq d_G(g'_2) \geq \ldots \geq d_G(g'_{n-r})$.

Proposition 3.1: Let k and n be two integers with $3 \leq k \leq n$, and let G be a connected threshold graph of order n. Let S be a set of distinct vertices of G such that $|S| = k$. Let g_i be the vertex in $S \cap V(C_r)$ with the minimum subscript, and g'_j be the vertex in $S \cap V(I_{n-r})$ with the maximum subscript.

(i) If $S \subseteq V(C_r)$, then $d_G(S) = k - 1$.
(ii) If $S \subseteq V(I_{n-r})$, then $d_G(S) = k$.
(iii) If $S \cap V(C_r) \neq \emptyset$, $S \cap V(I_{n-r}) \neq \emptyset$, and $g_i g'_j \in E(G)$, then $d_G(S) = k - 1$.
(iv) If $S \cap V(C_r) \neq \emptyset$, $S \cap V(I_{n-r}) \neq \emptyset$, and $g_i g'_j \notin E(G)$, then $d_G(S) = k$.

***Proof*:** From the structure of threshold graph G, (i) and (ii) can be easily seen. We only give the proof of (iii) and (iv). Let $S \cap V(C_r) = \{g_{i_1}, g_{i_2}, ..., g_{i_x}\}$ such that $i_1 \geq i_2 \geq ... \geq i_x$, and $S \cap V(I_{n-r}) = \{g'_{j_1}, g'_{j_2}, ..., g'_{j_{k-x}}\}$ such that $j_1 \geq j_2 \geq ... \geq j_{k-x}$. Then $g'_{i_1} = g_i$ and $g'_{j_{k-x}} = g'_j$. For (iii), since $g_i\, g'_j \in E(G)$, it follows that the tree induced by the edges in $\{g_{i_1}g'_{j_1}, g_{i_1}g'_{j_2}, ..., g_{i_1}g'_{j_{k-x}}\} \cup \{g_{i_1}g_{i_2}, g_{i_1}g_{i_3}, ..., g_{i_1}g_{i_x}\}$ is an S-Steiner tree in G, and hence $d_G(S) \leq k - 1$. So, we have $d_G(S) = k - 1$. For (iv), since $K_{1,n-1}$ is the spanning tree of G, it follows from Observation 1.2 that $sdiam_k(G) \leq sdiam_k(K_{1,n-1}) = k$. Since $g_i g'_j \notin E(G)$, it follows that $g_{i_1}g'_j$, $g_{i_2}g'_j$, ..., $g_{i_x}g'_j \notin E(G)$, and hence there is no S-Steiner tree of size $k - 1$ in G. Then $d_G(S) \geq k$, and hence $d_G(S) = k$. ❑

For Steiner diameter of threshold graphs, we have the following.

Proposition 3.2: Let k and n be two integers with $3 \leq k \leq n$, and let G be a connected threshold graph of order n. Let i be the subscript of the vertex in $V(C_r)$ such that $g_i g'_{n-r} \in E(G)$ but $g_{i+1}g'_{n-r} \notin E(G)$.

(i) If $3 \leq k \leq n - i$, then $sdiam_k(G) = k$.
(ii) If $n - i + 1 \leq k \leq n$, then $sdiam_k(G) = k - 1$.

***Proof*:** Recall that C_r and I_{n-r} are the clique and the maximum independent set of G, respectively, with $V(C_r) = \{g_1, g_2, \ldots, g_r\}$ and $V(I_{n-r}) = \{g'_1, g'_2, ..., g'_{n-r}\}$ such that $deg_G(g_1) \geq deg_G(g_2) \geq \ldots \geq deg_G(g_r)$ and $deg_G(g'_1) \geq deg_G(g'_2) \geq \ldots \geq deg_G(g'_{n-r})$. Since $K_{1,n-1}$ is the spanning tree of G, it follows from Observation 1.2 that $sdiam_k(G) \leq sdiam_k(K_{1,n-1}) = k$.

For (i), if $3 \leq k \leq n - r$, then we choose $S \subseteq V(I_{n-r}) \subseteq V(G)$ such that $|S| = k$. Since I_{n-r} is the maximum independent set of G, it follows that any S-Steiner tree must occupy at least one vertex in C_r, and hence $d_G(S) \geq k$. Then $sdiam_k(G) \geq d_G(S) = k$, and hence $sdiam_k(G) = k$. Suppose $n - r + 1 \leq k \leq n - i$. Choose $S \subseteq V(G) \backslash \{g_1, g_2, \ldots, g_i\}$ with $|S| = k$ such that $g'_{n-r} \in S$. Note that for each $j(i + 1 \leq j \leq r)$, $g_j g'_{n-r} \notin E(G)$. Since any S-Steiner tree must occupy at least one vertex in $\{g_1, g_2, \ldots, g_i\}$, it follows that $d_G(S) \geq k$, and hence $sdiam_k(G) \geq d_G(S) = k$. So, we have $sdiam_k(G) = k$.

For (ii), for any $S \subseteq V(G)$ such that $|S| = k$, since $n - i + 1 \leq k \leq n$, it follows that there exists a vertex $g_p \in \{g_1, g_2, \ldots, g_i\}$ such that $g_p \in S$. Let $S_1 = (S\cap V(C_r))\backslash g_p$ and $S_2 = S\cap V(I_{n-r})$. Then the tree induced by the edges in $E_{C_r}[g_p, S_1]\cup E_{I_{n-r}}[g_p, S_2]$ is an S-Steiner tree in G, and hence $d_G(S) \le k-1$. Then $sdiam_k(G) \le k-1$, and hence $sdiam_k(G) = k-1$. ❑

4. STEINER DIAMETER OF THE CORONA OF CONNECTED GRAPHS

In this section, let G and H be two graphs with $V(G) = \{g_1, g_2, ..., g_n\}$ and $V(H) = \{h_1, h_2, ..., h_m\}$, respectively. From the definition of corona graphs, $V(G\circ H) = V(G)\cup \{(g_i, h_j) \mid 1\le i\le n, 1\le j\le m\}$, where $\circ$ denotes the corona product operation. For $g\in V(G)$, we use $H(g)$ to denote the subgraph of $G\circ H$ induced by the vertex set $\{(g, h_j) \mid 1\le j\le m\}$. For fixed $i(1\le i\le n)$, we have $g_i(g_i, h_j)\in E(G\circ H)$ for each $j(1\le j\le m)$. Then $V(G\circ H) = V(G)\cup V(H(g_1))\cup V(H(g_2))\cup ... \cup V(H(g_n))$.

Theorem 4.1: Let k, m, and n be three integers with $3\le k\le n(m+1)$, and let G and H be two connected graphs with $V(G) = \{g_1, g_2, ..., g_n\}$ and $V(H) = \{h_1, h_2, ..., h_m\}$. Let S be a set of distinct vertices of $G\circ H$ such that $|S| = k$.

$$d_{G\circ H}(S) = d_G(S'_G) + k - t,$$

where $|S\cap V(G)| = t$, and S'_G is the maximum subset of $V(G)$ such that $S\cap(V(H(g))\cup\{g\}) \ne \emptyset$ for each $g\in S'_G$.

***Proof*:** Without loss of generality, we can assume $S'_G = \{g_1, g_2, ..., g_r\}$ is the maximum subset of $V(G)$ such that $S\cap(V(H(g_i))\cup\{g_i\}) = \emptyset$ for each $g\in S'_G$. Then $S\subseteq \cup_{i=1}^{r}(V(H(g_i))\cup\{g_i\})$. Clearly, $|S\cap\{g_1, g_2, ..., g_r\}| = |S\cap V(G) = t$. Without loss of generality, say $S\cap\{g_1, g_2, ..., g_r\} = \{g_1, g_2, ..., g_t\}$. Then since $t\le r$, $S\cap\{g_{t+1}, g_{t+2}, ..., g_r, g_{r+1}, ..., g_n\} = \emptyset$.

For each $i(1\le i\le r)$, we have $S\cap(V(H(g_i))\cup\{g_i\}) \ne \emptyset$. On one hand, we let T'_G be an S'_G-Steiner tree of size $d_G(S'_G)$ in G. Since $|S\cap V(G)| = t$, it follows that $|S\cap(\cup_{i=1}^{r}V(H(g_i)))| = k-t$. Let $S\cap(\cup_{i=1}^{r}V(H(g_i))) = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), ..., (g_{i_{k-t}}, h_{j_{k-t}})\}$, where $\{g_{i_1}, g_{i_2}, ..., g_{i_{k-t}}\}\subseteq\{g_1, g_2, ..., g_r\}$ and $\{h_{j_1}, h_{j_2}, ..., h_{j_{k-t}}\}\subseteq V(H)$. Then the tree induced by the edges in $E(T'_G)\cup\{g_{i_p}(g_{i_p}, h_{j_p})|1\le p\le k-t\}$ is an S-Steiner tree in $G\circ H$, and hence $d_{G\circ H}(S)\le d_G(S'_G) + k - t$.

On the other hand, since $S\cap(V(H(g_i))\cup\{g_i\}) \ne \emptyset$ for each $g_i\in S'_G$, it follows from the structure of $G\circ H$ that any S-Steiner tree T must contains all vertices in $S'_G = \{g_1, g_2, ..., g_r\}$. Let T' be the minimal subtree of T connecting S'_G. Then $e(T)\ge d_G(S'_G)$. For all vertices in $S\cap(\cup_{i=1}^{r}V(H(g_i)))$, we need at least $k-t$ edges connecting the vertices in $S\cap(\cup_{i=1}^{r}V(H(g_i)))$ to S'_G. So, we have $d_{G\circ H}(S)\ge e(T') + k - t\ge d_G(S'_G) + k - t$.

From the above argument, we conclude that $d_{G\circ H}(S) = d_G(S'_G) + k - t$, as desired. ❑

For Steiner diameter of corona graphs, we have the following.

Proposition 4.1: Let k, n, and m be integers with $3\le k\le n(m+1)$, and let G and H be connected graphs with n and m vertices, respectively.

(i) If $3\le k\le n$, then $sdiam_k(G\circ H) = sdiam_k(G) + k$.

(ii) If $n + 1 \leq k \leq mn$, then $sdiam_k(G \circ H) = n - 1 + k$.
(iii) If $mn + 1 \leq k \leq (m + 1)n$, then $sdiam_k(G \circ H) = n - 1 + mn$.

***Proof*:** (i) For the upper bound, from the definition of $sdiam_k(G \circ H)$, there exists a vertex subset $S \subseteq V(G \circ H)$ with $|S| = k$ such that $d_{G \circ H}(S) = sdiam_k(G \circ H)$. Let $S = \{(g_1, h_1), (g_2, h_2), ..., (g_k, h_k)\}$, and let $S'_G \subseteq \{g_1, g_2, ..., g_k\}$. From Theorem 4.1, we have $sdiam_k(G \circ H) = d_{G \circ H}(S) = d_G(S'_G) + k - t \leq sdiam_k(G) + k - t \leq sdiam_k(G) + k$. For the lower bound, we choose $S = \{(g_1, h_1), (g_2, h_1), ..., (g_k, h_1)\}$ such that $d_G(S_G) = sdiam_k(G)$, where $S_G = \{g_1, g_2, ..., g_k\}$. Then any S-Steiner tree T must contain all vertices in $S_G = \{g_1, g_2, ..., g_k\}$, and we also need a subtree T' of T of size at least $d_G(S_G)$. For each vertex in S, we need at least one edge to connect it to T', and hence $sdiam_k(G \circ H) \geq d_{G \circ H}(S) \geq d_G(S_G) + k = sdiam_k(G) + k$. We conclude that $sdiam_k(G \circ H) = sdiam_k(G) + k$.

(ii) For the upper bound, from the definition of $sdiam_k(G \circ H)$, there exists a vertex subset $S \subseteq V(G \circ H)$ with $|S| = k$ such that $d_{G \circ H}(S) = sdiam_k(G \circ H)$. Let $S = \{(g_1, h_1), (g_2, h_2), ..., (g_k, h_k)\}$, and let $S \subseteq \{g_1, g_2, ..., g_k\}$. From Theorem 4.1, we have $sdiam_k(G \circ H) = d_{G \circ H}(S) = d_G(S') + k - t \leq n - 1 + k - t \leq n - 1 + k$. For the lower bound, since $n + 1 \leq k \leq mn$, we choose $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_1}), ..., (g_{i_k}, h_{j_k})\}$ such that $V(G \subseteq \{g_{i_1}, g_{i_2}, ..., g_{i_k}\}$. Let $S_G = \{g_{i_1}, g_{i_2}, ..., g_{i_k}\}$. Then any S-Steiner tree T must contain all vertices in $S_G = V(G)$, and we need a subtree T' of T of size at least $n - 1$. For each vertex in S, we need at least one edge to connect it to T', and hence $sdiam_k(G \circ H) \geq d_{G \circ H}(S) \geq n - 1 + k$. We conclude that $sdiam_k(G \circ H) = n - 1 + k$.

(iii) Since $|V(G \circ H)| = mn + n$, it follows that $sdiam_k(G \circ H) \leq n - 1 + mn$. Furthermore, since $mn + 1 \leq k \leq (m + 1)n$, we choose $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_1}), ..., (g_{i_k}, h_{j_k})\}$ such that $V(G) \subseteq \{g_{i_1}, g_{i_2}, ..., g_{i_k}\}$ and $V(H) \subseteq \{h_{j_1}, h_{j_2}, ..., h_{j_k}\}$, that is, $V(H(g_1)) \cup V(H(g_2)) \cup ... \cup V(H(g_n)) \subseteq S$. Then any S-Steiner tree T must contain all vertices in $V(G \circ H)$, and hence $sdiam_k(G \circ H) \geq n - 1 + mn$. We conclude that $sdiam_k(G \circ H) = n - 1 + mn$. ❑

5. STEINER DIAMETER OF THE CLUSTER OF CONNECTED GRAPHS

In this section, let G and H be two graphs with $V(G) = \{g_1, g_2, ..., g_n\}$ and $V(H) = \{h_1, h_2, ..., h_m\}$, respectively. From the definition of cluster, $V(G \odot H) = \{(g_i, h_j) | 1 \leq i \leq n, 1 \leq j \leq m\}$, where $\odot$ denotes the cluster product operation. For $g \in V(G)$, we use $H(g)$ to denote the subgraph of $G \odot H$ induced by the vertex set $\{(g, h_j) \mid 1 \leq j \leq m\}$. Without loss of generality, we assume (g_i, h_1) is the root of $H(g_i)$ for each $g_i \in V(G)$. Let $G(h_1)$ be the graph induced by the vertices in $\{(g_i, h_1) | 1 \leq i \leq n\}$. Clearly, $G(h_1) \simeq G$, and $V(G \odot H) = V(H(g_1)) \cup V(H(g_2)) \cup ... \cup V(H(g_n))$.

Theorem 5.1: Let k, m, n be three integers with $3 \leq k \leq nm$, and let G, H be two connected graphs with $V(G) = \{g_1, g_2, ..., g_n\}$ and $V(H) = \{h_1, h_2, ..., h_m\}$. Let $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), ..., (g_{i_k}, h_{j_k})\}$ be a set of distinct vertices of $G \odot H$. Let $S_G = \{g_{i_1}, g_{i_2}, ..., g_{i_k}\}$ and $S_H = \{h_{j_1}, h_{j_2}, ..., h_{j_k}\}$.

(i) If $S \subseteq V(G(h_1))$, then $d_{G \odot H}(S) = d_G(S_G)$.

(ii) If there exists some $H(g_i)(1 \le i \le n)$ such that $S \subseteq V(H(g_i))$, then $d_{G \odot H}(S) = d_H(S_H)$.
(iii) If there is no $H(g_i)(1 \le i \le n)$ such that $S \subseteq V(H(g_i))$, then

$$d_G(S'_G) + k - t \le d_{G \odot H}(S) \le r \cdot d_H(S'_H) + d_G(S'),$$

where $S'_H = S_H$ if $h_1 \in S_H$ and $S'_H = S_H \cup \{h_1\}$ otherwise $|S \cap V(G(h_1))| = t$, $|S'_G| = r$, and S'_G is the maximum subset of $V(G)$ such that $S \cap V(H(g)) \neq \emptyset$ for each $g \in S'_G$.

Moreover, the upper and lower bounds in (iii) are sharp.

***Proof*:** We only give the proof of (iii), since (i) and (ii) are both easily seen. Without loss of generality, we can assume $S'_G = \{g_1, g_2, ..., g_r\}$ is the maximum subset of $V(G(h_1))$ such that $S \cap V(H(g_i)) \neq \emptyset$ for each $g_i \in S'_G$. Then $S \subseteq \cup_{i=1}^{r} V(H(g_i))$, and $S \cap V(H(g_i)) = \emptyset$ for each $i(r + 1 \le i \le n)$. Clearly, $|S \cap \{g_1, g_2, ..., g_r\}| = |S \cap V(G(h_1))| = t$. Without loss of generaliy, $S \cap \{g_1, g_2, ..., g_r\} = \{g_1, g_2, ..., g_t\}$. Then since $t \le r$, $S \cap \{g_{t+1}, g_{t+2}, ..., g_r\} = \emptyset$ and $S \cap \{g_{r+1}, g_{r+2}, ..., g_n\} = \emptyset$.

We first give the lower bound. Since $S \cap V(H(g_i)) \neq \emptyset$ for each $g_i \in S'_G$, it follows from the structure of $G \odot H$ that any S-Steiner tree T must contains any vertices in $S'_G = \{(g_i, h_1) \mid 1 \le i \le r\}$. Let T' be the minimal subtree of T connecting S'. Then $e(T') \ge d_G(S'_G)$. For all vertices in $S \cap (\cup_{i=2}^{r} | V(H(g_i)))$, we need at least $k - t$ edges connecting them to S'. So, we have $d_{G \odot H}(S) \ge e(T') + k - t = d_G(S'_G) + k - t$, as desired.

Next, we give the proof of the upper bound. If $h_1 \in S_H$, then without loss of generality, let $h_1 = h_{j_1}$. Note that there is an S'_G-Steiner tree T of size $d_G(S'_G)$ in $G(h_1)$. Since there is an S_H-Steiner tree of size $d_H(S_H)$ in H, it follows that there exists a Steiner tree of size $d_H(S_H)$ connecting $\{(g_1, h_{j_1}), (g_1, h_{j_2}), ..., (g_1, h_{j_k})\}$ in $H(g_1)$, say $T(g_1)$. For each $i(2 \le i \le k)$, let $T(g_i)$ be the tree in $H(g_i)$ corresponding to $T(g_1)$ in $H(g_1)$. Note that $T(g_i)(1 \le i \le k)$ is the Steiner tree of size $d_H(S_H)$ connecting $\{(g_i, h_{j_1}), (g_i, h_{j_2}), ..., (g_i, h_{j_k})\}$ in $H(g_i)$. One can see that $(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), ..., (g_{i_k}, h_{j_k}) \in \cup_{i=2}^{q} | V(T(g_i))$. Furthermore, the subgraph induced by the edges in $(\cup_{i=1}^{q} E(T(g_i)) \cup E(T))$ is an S-Steiner tree, and hence $d_{G \odot H}(S) \le d_G(S'_G) + r d_H(S_H)$. If $h_1 \notin S_H$, then there is an S'_G-Steiner tree T of size $d_G(S'_G)$ in $G(h_1)$. Since there is an $S_H \cup \{h_1\}$-Steiner tree of size $d_H(S_H \cup \{h_1\})$ in H, it follows that there exists a Steiner tree of size $d_H(S_H \cup \{h_1\})$ connecting $\{(g_1, h_1)\} \cup \{(g_1, h_{j_1}), (g_1, h_{j_2}), ..., (g_1, h_{j_k})\}$ in $H(g_1)$, say $T(g_1)$. For each $i(2 \le i \le k)$, let $T(g_i)$ be the tree in $H(g_i)$ corresponding to $T(g_1)$ in $H(g_1)$. Note that $T(g_i)(1 \le i \le k)$ is the Steiner tree of size $d_H(S_H)$ connecting $\{(g_i, h_1)\} \cup \{(g_i, h_{j_1}), (g_i, h_{j_2}), ..., (g_i, h_{j_k})\}$ in $H(g_i)$. One can see that $(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_2}), ..., (g_{i_k}, h_{j_k})$ $V(T(g_i))$. Furthermore, the subgraph induced by the edges in $(\cup_{i=1}^{q} E(T(g_i))) \cup E(T))$ is an S-Steiner tree, and hence $d_{G \vee H}(S) \le d_G(S'_G) + r d_H(S_H \cup \{h_1\})$. The result follows. ❑

To show the sharpness of the above lower and upper bounds, we consider the following examples.

Example 1: Let $G = P_n = g_1 g_2 \ldots g_n$ and $H = P_m = h_1 h_2 \ldots h_m$ with $3 \le k \le mn$. Note that $H(g_i) \cong P_m$ for each $g_i (1 \le i \le n)$, and $G(h_1) \cong P_n$. For $h_1 \notin S_H$, if $k \le n$, then we choose $S = \{(g_1, h_m), (g_2, h_m), ..., (g_{k-1}, h_m)\} \cup \{(g_n, h_m)\}$. Then $r = k$, $d_H(S_H \cup \{h_1\}) = m - 1$, $d_G(S'_G) = n - 1$. Since the tree induced by the edges in $E(G(h_1)) \cup E(H(g_1)) \cup E(H(g_2)) \cup ... \cup E(H(g_{k-1})) \cup E(H(g_n))$ is the unique S-Steiner tree, it follows that $d_{G \odot H}(S) \ge k(m - 1) + (n - 1)$.

From Theorem 5.1, $d_{G\odot H}(S) \leq rd_H(S_H \cup \{h_1\}) + d_G(S'_G) = k(m-1) + (n-1)$. So, the upper bound for $h_1 \notin S_H$ is sharp. For $h_1 \in S_H$, if $k \leq n$, then we choose $S = \{(g_1, h_1), (g_2, h_1), ..., (g_k, h_1)\}$. Then $r = k$, $d_H(S_H) = 0$, $d_G(S'_G) = k-1$. Then $d_{G\odot H}(S) \geq k-1$. From Theorem 5.1, $d_{G\odot H}(S) \leq rd_H(S_H) + d_G(S'_G) = k-1$. So, the upper bound for $h_1 \in S_H$ is sharp.

Example 2: Let $G = P_n = g_1g_2..., g_n$ and $H = K_m$ with $3 \leq k \leq mn$, where $V(H) = \{h_1, h_2, ..., h_n\}$. Note that $H(g_i) \cong K_m$ for each $g_i (1 \leq i \leq n)$, and $G(h_1) \cong P_n$. Choose $S = \{(g_1, h_m), (g_2, h_m), ..., (g_{k-1}, h_m)\} \cup \{(g_n, h_m)\} (m \geq 2)$. Then $d_G(S'_G) = n-1$ and $t = 0$, and hence $d_{G\odot H}(S) \geq n-1+k$. Clearly, the tree induced by the edges in $E(G(h_1)) \cup E_{G\odot H}[V(G(h_1)), S]$ is an S-Steiner tree in $G\odot H$, and hence $d_{G\odot H}(S) \leq n-1+k$. So, we have $d_{G\odot H}(S) = n-1+k$, which implies that the lower bound is sharp.

Corollary 5.1 Let k, m, n be three integers with $3 \leq k \leq nm$, and let G, H be two connected graphs with $V(G) = \{g_1, g_2, ..., g_n\}$ and $V(H) = \{h_1, h_2, ..., h_m\}$. Let (g_i, h_1) be the root of $H(g_i)$ for each $g_i (1 \leq i \leq n)$. Let S be a set of distinct vertices of $G\odot H$ such that $|S| = k$. Then

$$d_{G\odot H}(S) \leq rd_H(S_H \cup \{h_1\}) + d_G(S'_G),$$

where $|S'_G| = r$, and S'_G is the maximum subset of $V(G)$ such that $S \cap V(H(g)) \neq \emptyset$ for each $g \in S'_G$.

For Steiner diameter of cluster graphs, we have the following.

Proposition 5.1: Let k, n, m be two integers with $3 \leq k \leq nm$, and let G, H be two connected graphs with n, m vertices, respectively.

(i) If $m > n$ and $3 \leq k \leq n$, then
$$sdiam_k(G) + k \leq sdiam_k(G\odot H) \leq k \cdot sdiam_{k+1}(H) + sdiam_k(G).$$
(ii) If $m > n$ and $n+1 \leq k \leq m-1$, then
$$n-1+k \leq sdiam_k(G\odot H) \leq n \cdot sdiam_{k+1}(H) + n-1.$$
(iii) If $m > n$ and $m \leq k \leq nm-n$, then
$$n-1+k \leq sdiam_k(G\odot H) \leq mn-1.$$
(iv) If $m > n$ and $nm-n \leq k \leq nm$, then $sdiam_k(G\odot H) = nm-1$.
(v) If $m \leq n$ and $3 \leq k < m$, then
$$sdiam_k(G) + k \leq sdiam_k(G\odot H) \leq k \cdot sdiam_{k+1}(H) + sdiam_k(G).$$
(vi) If $m \leq n$ and $m \leq k \leq n$, then
$$sdiam_k(G) + k \leq sdiam_k(G\odot H) \leq k(m-1) + sdiam_k(G).$$
(vii) If $m \leq n$ and $n < k \leq mn-n$, then
$$n-1+k \leq sdiam_k(G\odot H) \leq mn-1.$$
(viii) If $m \leq n$ and $mn-n < k \leq mn$, then $sdiam_k(G\odot H) = mn-1$.

Moreover, the upper and lower bounds are sharp.

***Proof*:** (i) For the upper bound, from the definition of $sdiam_k(G\odot H)$, there exists a vertex subset $S \subseteq V(G\odot H)$ with $|S| = k$ such that $d_{G\odot H}(S) = sdiam_k(G\odot H)$. Let $S = \{(g_1, h_1), (g_2,$

h_2), ..., (g_k, h_k)}, and let $S_G = \{g_1, g_2, ..., g_k\}$ and $S_H = \{h_1, h_2, ..., h_k\}$. From Theorem 5.1, we have $sdiam_k(G \odot H) = d_{G \odot H}(S) \leq rd_H(S_H \cup \{h_1\}) + d_G(S'_G) \leq k \cdot sdiam_{k+1}(H) + sdiam_k(G)$. For the lower bound, since $k \leq n$, we choose $S = \{(g_1, h_m), (g_2, h_m), ..., (g_k, h_m)\}(m \geq 2)$ such that $d_G(S_G) = sdiam_k(G)$, where $S_G = \{g_1, g_2, ..., g_k\}$. Then any S-Steiner tree T must contain all vertices in $\{(g_1, h_1), (g_2, h_1), ..., (g_k, h_1)\}(m \geq 2)$, and we also need a subtree T' of T of size at least $d_G(S_G)$ in $G(h_1)$. For each vertex in S, we need at least one edge to connect it to T', and hence $sdiam_k(G \odot H) \geq d_{G \odot H}(S) \geq d_G(S_G) + k = sdiam_k(G) + k$. We conclude that $sdiam_k(G) + k \leq sdiam_k(G \odot H) \leq k \cdot sdiam_{k+1}(H) + sdiam_k(G)$.

(ii) For the upper bound, from the definition of $sdiam_k(G \odot H)$, there exists a vertex subset $S \subseteq V(G \odot H)$ with $|S| = k$ such that $d_{G \odot H}(S) = sdiam_k(G \odot H)$. Let $S = \{(g_1, h_1), (g_2, h_2), ..., (g_k, h_k)\}$, and let $S_G = \{g_1, g_2, ..., g_k\}$ and $S_H = \{h_1, h_2, ..., h_k\}$. From Theorem 5.1, we have $sdiam_k(G \odot H) = d_{G \odot H}(S) \leq rd_H(S_H \cup \{h_1\}) + d_G(S') \leq n \cdot sdiam_{k+1}(H) + n - 1$, where S'_G is the maximum subset of $V(G)$ such that $S \cap V(H(g)) = \emptyset$ for each $g \in S'_G$. For the lower bound, since $n + 1 \leq k \leq m - 1$, we choose $S = \{(g_1, h_m), (g_2, h_m), ..., (g_k, h_m)\}(m \geq 2)$ such that $V(G) \subseteq S_G$, $S_G = \{g_1, g_2, ..., g_k\}$. Then any S-Steiner tree T must contain all vertices in $\{(g_1, h_1), (g_2, h_2), ..., (g_n, h_1)\}(m \geq 2)$, and we need a subtree T' of T of size $n - 1$ {in $G(h_1)$. For each vertex in S, we need at least one edge to connect it to T', and hence $sdiam_k(G \odot H) \geq d_{G \odot H}(S) \geq n - 1 + k$. We conclude that $n - 1 + k \leq sdiam_k (G \odot H) \leq n \cdot sdiam_{k+1}(H) + n - 1$.

(iii) For the upper bound, from the definition of $sdiam_k(G \odot H)$, there exists a vertex subset $S \subseteq V(G \odot H)$ with $|S| = k$ such that $d_{G \odot H}(S) = sdiam_k(G \odot H)$. Let $S = \{(g_1, h_1), (g_2, h_2), ..., (g_k, h_k)\}$, and let $S_G = \{g_1, g_2, ..., g_k\}$ and $S_H = \{h_1, h_2, ..., h_k\}$. From Theorem 5.1, we have $sdiam_k(G \odot H) = d_{G \odot H}(S) \leq rd_H(S_H \cup \{h_1\}) + d_G(S') \leq n(m - 1) + n - 1 = nm - 1$, where S is the maximum subset of $V(G)$ such that $S \cap V(H(g)) \neq \emptyset$ for each $g \in S'$. For the lower bound, since $m \leq k \leq nm - n$, we choose $S = \{(g_1, h_m), (g_2, h_m), ..., (g_k, h_m)\}(m \geq 2)$ such that $V(G) \subseteq S_G$, where $S_G = \{g_1, g_2, ..., g_k\}$. Then any S-Steiner tree T must contain all vertices in $\{(g_1, h_1), (g_2, h_1), ..., (g_n, h_1)\}$, and we need a subtree T' of T of size $n - 1$ in $G(h_1)$. For each vertex in S, we need at least one edge to connect it to T', and hence $sdiam_k(G \odot H) \geq d_{G \odot H}(S) \geq n - 1 + k$. We conclude that $n - 1 + k \leq sdiam_k(G \odot H) \leq mn - 1$.

(iv) Since $|V(G \odot H)| = mn$, it follows that $sdiam_k(G \odot H) \leq mn - 1$. Furthermore, since $nm - n \leq k \leq nm$, we choose $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_1}), ..., (g_{i_k}, h_{j_k})\}$ such that $V(G) \backslash \{g_1\} \subseteq \{g_{i_1}, g_{i_2}, ..., g_{i_k}\}$ and $V(H) \backslash \{h_1\} \subseteq \{h_{j_1}, h_{j_2}, ..., h_{j_k}\}$, that is, $V(H(g_2)) \cup V(H(g_3)) \cup ... \cup V(H(g_n)) \subseteq S$. Then any S-Steiner tree T must contain all vertices in $V(G)$, and hence $sdiam_k(G \odot H) \geq nm - 1$. So, we have $sdiam_k(G \odot H) = nm - 1$.
(v) Since $3 \leq k < m \leq n$, the proof of this case is similar to (i), and thus omitted.
(vi) For the upper bound, from the definition of $sdiam_k(G \odot H)$, there exists a vertex subset $S \subseteq V(G \odot H)$ with $|S| = k$ such that $d_{G \odot H}(S) = sdiam_k(G \odot H)$. Let $S = \{(g_1, h_1), (g_2, h_2), ..., (g_k, h_k)\}$, and let $S_G = \{g_1, g_2, ..., g_k\}$ and $S_H = \{h_1, h_2, ..., h_k\}$.

From Theorem 5.1, we have $sdiam_k(G \odot H) = d_{G \odot H}(S) \leq rd_H(S_H \cup \{h_1\}) + d_G(S') \leq k(m - 1) + sdiam_k(G)$, where S'_G is the maximum subset of $V(G)$ such that $S \cap V(H(g)) \neq \emptyset$ for each $g \in S'_G$. For the lower bound, since $m \leq k \leq n$, we choose $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_1}), ...,$

$(g_{i_1}, h_{j_k})\}$ such that $V(H)\backslash\{h_1\} \subseteq S_H$ and $d_G(S_G) = sdiam_k(G)$, where $S_G = \{g_{i_1}, g_{i_1}, ..., g_{i_k}\}$. Then any S-Steiner tree T must contain all vertices in $\{(g_1, h_1), (g_2, h_1), ..., (g_k, h_1)\}$, and we need a subtree T' of T of size $d_G(S_G) = sdiam_k(G)$ in $G(h_1)$. For each vertex in S, we need at least one edge to connect it to T', and hence $sdiam_k(G \odot H) \geq d_{G\odot H}(S) \geq sdiam_k(G) + k$. We conclude that $sdiam_k(G) + k \leq sdiam_k(G \odot H) \leq k(m - 1) + sdiam_k(G)$.

(vii) Since $|V(G \odot H)| = mn$, it follows that $sdiam_k(G \odot H) \leq mn - 1$. Furthermore, since $m \leq n < k \leq mn - n$, we choose $S = \{(g_{i_1}, h_{j_1}), (g_{i_2}, h_{j_1}), ..., (g_{i_1}, h_{j_k})\}$ such that $V(H)\backslash\{h_1\} \subseteq S_H$ and $V(G) \subseteq S_G$, where $S_G = \{g_{i_1}, g_{i_1}, ..., g_{i_k}\}$. Then any S-Steiner tree T must contain all vertices in $\{(g_1, h_1), (g_2, h_1), ..., (g_n, h_1)\}$, and we need a subtree T' of T of size $n - 1$ in $G(h_1)$. For each vertex in S, we need at least one edge to connect it to T', and hence $sdiam_k(G \odot H) \geq d_{G\odot H}(S) \geq n - 1 + k$. We conclude that $n - 1 + k \leq sdiam_k(G \odot H) \leq mn - 1$.

(viii) For $m \leq n$ and $mn - n < k \leq mn$, the proof of this case is similar to (iv), and thus omitted. ❑

To show the sharpness of the above upper bounds, we consider the following example.

Example 3: Let $G = P_n = g_1g_2, ..., g_n$ and $H = P_m = h_1h_2..., h_m$ with $3 \leq k \leq mn$. Note that $H(g_i) \cong P_m$ for each $g_i(1 \leq i \leq n)$, and $G(h_1) \cong P_n$.

For (i), we have $3 \leq k \leq n < m$, $sdiam_k(G) = n - 1$, and $sdiam_{k+1}(H) = m - 1$. From (i) of Proposition 5.1, $sdiam_k(G \odot H) \leq ksdiam_{k+1}(H) + sdiam_k(G) = k(m - 1) +(n - 1)$. Choose $S = \{(g_1, h_m), (g_2, h_m), ..., (g_{k-1}, h_m)\} \cup \{(g_n, h_m)\}$. Then the tree induced by the edges in $E(G(h_1)) \cup E(H(g_1)) \cup E(H(g_2)) \cup ... \cup E(H(g_{k-1})) \cup E(H(g_n))$ is the unique S-Steiner tree, and hence $sdiam_k(G \odot H) = d_{G\odot H}(S) \geq k(m - 1) +(n - 1)$. So, we have $sdiam_k(G \odot H) = k(m - 1) +(n - 1)$, which implies that the upper bound in (i) is sharp. Similarly, the upper bound in (v) is sharp.

For (ii), we have $m > n$, $n + 1 \leq k \leq m - 1$, and $sdiam_{k+1}(H) = m - 1$. From (i) of Proposition 5.1, $sdiam_k(G \odot H) \leq nsdiam_{k+1}(H) + n - 1 = n(m - 1) + n - 1 = mn - 1$. Choose $S \subseteq V(G \odot H)$ with $|S| = k$ such that $\{(g_1, h_m), (g_2, h_m), ..., (g_n, h_m)\} \subseteq S$. Then the tree induced by the edges in $E(G \odot H)$ is the unique S-Steiner tree, and hence $sdiam_k(G \odot H) = d_{G\odot H}(S) \geq mn - 1$. So, we have $sdiam_k(G \odot H) = mn - 1$, which implies that the upper bound in (ii) is sharp. Similarly, the upper bound in (vi) is sharp.

For (iii), we have $n < m \leq k \leq nm - n$. From (iii) of Proposition 5.1, we have $sdiam_k(G \odot H) \leq mn - 1$. Choose $S \subseteq V(G \odot H)$ with $|S| = k$ such that $\{(g_1, h_m), (g_2, h_m), ..., (g_n, h_m)\} \subseteq S$. Then the tree induced by the edges in $E(G \odot H)$ is the unique S-Steiner tree, and hence $sdiam_k(G \odot H) = d_{G\odot H}(S) \geq mn - 1$. So, we have $sdiam_k(G \odot H) = mn - 1$, which implies that the upper bound in (iii) is sharp. Similarly, the upper bound in (vii) is sharp.

To show the sharpness of the above lower bounds, we consider the following example.

Example 4: Let $G = P_n = g_1g_2...g_n$ and $H = K_m$ with $3 \leq k \leq mn$, where $V(H) = \{h_1, h_2, ..., h_n\}$. Note that $H(g_i) \cong K_m$ for each $g_i(1 \leq i \leq n)$, and $G(h_1) \cong P_n$. For (i)-(iii) and (v)-(vii), we have $3 \leq k \leq mn - n$, and $sdiam_k(G \odot H) \geq n - 1 + k$. For any $S \subseteq V(G \odot H)$ and $|S| = k$, the tree induced by the edges in $E(G(h_1)) \cup E_{G \odot H}[V(G(h_1)), S \backslash V(G(h_1))]$ is an S-Steiner tree in $G \odot H$, and hence $d_{G \odot H}(S) \geq n - 1 + k$. From the arbitrariness of S, we have $sdiam_k(G \odot H) = n - 1 + k$, which implies that the lower bounds in (i)-(iii) and (v)-(vii) are sharp.

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