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Power Domination in Honeycomb Meshes*

KUO-HUA KAO[1], JOU-MING CHANG[2], YUE-LI WANG[3,†],
SHUO-HONG XU[4] AND JUSTIE SU-TZU JUAN[1]
[1]*Department of Computer Science and Information Engineering*
National Chi Nan University
Nantou, 545 Taiwan
[2]*Institute of Information and Decision Sciences*
National Taipei College of Business
Taipei, 100 Taiwan
[3]*Department of Information Management*
National Taiwan University of Science and Technology
Taipei, 106 Taiwan
[4]*Data Processing Unit*
Directorate General of Highway
Ministry of Transportation and Communication
Taipei, 100 Taiwan

The power domination problem is to find a minimum placement of phase measurement units (PMUs) for observing the whole electric power system represented by a graph G. The number of such a minimum placement of PMUs is called the power domination number of G and is denoted by $\gamma_p(G)$. Finding $\gamma_p(G)$ of an arbitrary graph is known to be NP-complete. In this paper, we deal with the power domination problem on honeycomb meshes. For a t-dimensional honeycomb mesh HM_t, we show that $\gamma_p(HM_t) = \lceil 2t/3 \rceil$. In particular, we present an $O(t)$-time algorithm as the placement scheme.

Keywords: algorithms, power domination, phase measurement units, honeycomb meshes, spread domination

1. INTRODUCTION

As an important application in energy management, electric power companies gather available data by devices called *phase measurement units* (abbreviated as PMUs) to continually monitor and estimate their system's state defined by a set of state variables (such as bus voltage magnitudes at loads and machine phase angles at generators [1]). To achieve the high accuracy in this estimation, a solution requires the system network to be observable. A system is said to be *observed* if all of its state variables are inspected by a set of PMUs. Because of the high cost of PMUs, a well-designed placement can possibly make the whole system observable using fewer PMUs, and thus reduce the overall cost. Therefore, designing a satisfactory placement scheme of PMUs has become an important issue and is widely studied in [2-7].

An electric power system is usually represented by an undirected graph $G = (V, E)$, where V is a set of vertices consisting of all electric nodes of the system, and E is a set of

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† Corresponding author: ylwang@cs.ntust.edu.tw

edges consisting of all transmission lines joining electric nodes. As a variation of the well-known graph domination problem (see monographs [8, 9] and other related works [10, 11]), Haynes *et al.* [1] considered a graph theoretical representation of the power system monitoring problem by using three observation rules. According to the rules, a PMU measures the state variable of the vertex at which it is placed and observes its incident edges and their end-vertices. Brueni [12], Kneis *et al.* [13] and Zhao *et al*. [14] independently pointed out that all vertices and edges of a graph G are observed if and only if all vertices of G are observed. Thus, there is a way to simplify the problem description by using two rules instead of the original rules as follows:

Observation Rule 1 (abbreviated as OR1): A PMU on a vertex v observes v and all its neighbors.

Observation Rule 2 (abbreviated as OR2): If an observed vertex u has only one unobserved neighbor v, then v becomes observed as well.

For a graph $G = (V, E)$, a set $P \subseteq V$ is said to be a *power dominating set* (abbreviated as PDS) of G if every vertex of G is observed by P (*i.e.*, all vertices of V are observed either by OR1 initially or by OR2 recursively). A PDS is *minimum* if it has the minimum size among all power dominating sets of G. The *power domination number* of G, denoted by $\gamma_p(G)$, is the cardinality of a minimum PDS of G, and the *power domination problem* is a problem for finding $\gamma_p(G)$. Haynes *et al.* [1] showed that the power domination problem is closely related to the classical graph domination problem and is NP-complete even when restricted to some special classes of graphs such as bipartite graphs or chordal graphs. For more recent results related to the power domination on graphs, we refer the reader to [1, 3, 13-22]. In particular, Dorbec *et al.* [17] have devoted themselves to the research of power domination on product graphs which include grids as a special case. To the best of our knowledge, there is no further investigation of the power domination problem related to any variation of grids except [17, 18]. In this paper, we study the power domination problem on honeycomb meshes and provide an algorithm to obtain a minimum PDS, where the time complexity of our algorithm is proportional to the size of such a PDS.

Honeycomb meshes are defined as follows: One hexagon is a honeycomb mesh of size one, denoted HM_1. The honeycomb mesh HM_2 of size two is obtained by adding six hexagons to the boundary edges of HM_1. Inductively, a honeycomb mesh HM_t of size t is obtained from HM_{t-1} by adding a layer of hexagons around the boundary of HM_{t-1}. Here we use the coordinate system introduced by Stojmenovic [23]. Let x-, y- and z-axes start at the center of a honeycomb mesh and be parallel to the three edge directions, respectively (see Fig. 1). A honeycomb mesh is a bipartite graph and its vertices can be labeled by using integer triples (x, y, z) such that $1 - t \le x, y, z \le t$ and $1 \le x + y + z \le 2$. Two vertices (x, y, z) and (x', y', z') are adjacent if and only if $|x - x'| + |y - y'| + |z - z'| = 1$. For $HM_t = (V_1 \cup V_2, E)$, let $V_i = \{(x, y, z) \mid x + y + z = i\}$, $i = 1, 2$ (*i.e.*, V_1 and V_2 denote the sets of white vertices and black vertices in honeycomb mesh, respectively). Thus, if $(x, y, z) \in V_1$, then its adjacent neighbors are $(x + 1, y, z)$, $(x, y + 1, z)$, and $(x, y, z + 1)$. On the other hand, if $(x, y, z) \in V_2$, then its adjacent neighbors are $(x - 1, y, z)$, $(x, y - 1, z)$, and $(x, y, z - 1)$.

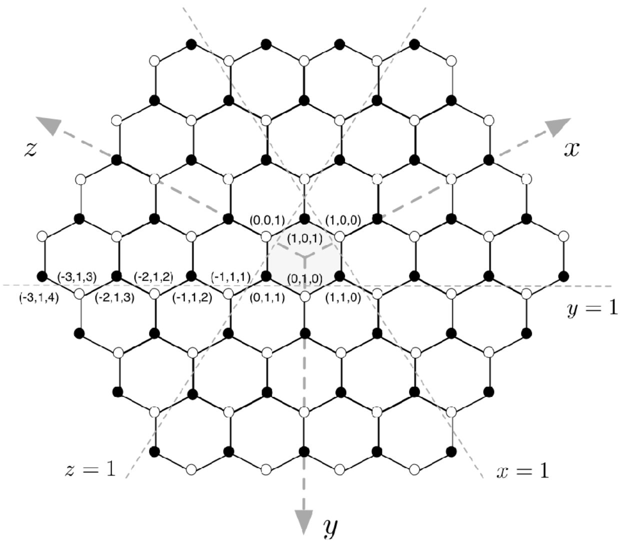


Fig. 1. A honeycomb mesh HM_4.

2. A LOWER BOUND OF THE POWER DOMINATION NUMBER ON HONEYCOMB MESHES

For a vertex u in a graph $G = (V, E)$, let $N(u) = \{v \in V \mid (u, v) \in E\}$ and $N[u] = N(u) \cup \{u\}$. This naturally generalizes to $N(U) = \cup_{u \in U} N(u)$ and $N[U] = N(U) \cup U$ for $U \subset V$. Let $\Delta_G = \max_{u \in U} |N(u)|$ denote the maximum degree of G and $d_G(u, v)$ denote the length of a shortest path from u to v in G, where we omit the subscript G if it is clear from context. From the observation rules defined in the previous section, for $S \subseteq V$, we write $R_1(S)$ and $R_2(S)$ as the sets of observed vertices by S, respectively, applying OR1, and OR2 recursively. Clearly, $R_1(S) = N[S]$ and $S \subseteq R_2(S)$. A set $S \subseteq V$ is said to be a *spread dominating set* (abbreviated as SDS) of G if every vertex of G is observed by S using only OR2 recursively (*i.e.*, $R_2(S) = V$). The *spread domination number* of G, denoted by $\gamma_s(G)$, is the minimum cardinality of an SDS of G. Note that if $S \subseteq S' \subseteq V$ and S is an SDS of G, then so is S'. For a bipartite graph G with partite sets V_1 and V_2, an SDS of G is said to be *bias* (abbreviated as bias-SDS) if it contains either V_1 or V_2 as a subset. In the rest of this paper, due to the symmetry of V_1 and V_2 in HM_t, we consider a bias-SDS containing all vertices of V_2 to establish some relevant properties. Also, for notational convenience, we define $R(S) = R_2(V_2 \cup S) \cap V_1$ for a set $S \subseteq V_1$. Hence, a set $S \subseteq V_1$ together with V_2 is a bias-SDS if and only if $R(S) = V_1$. The following lemma shows that we can easily obtain a lower bound of $\gamma_p(G)$ by means of a lower bound of $\gamma_s(G)$.

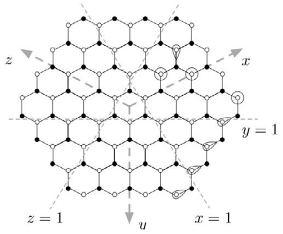
Lemma 1 For any graph $G = (V, E)$, $\gamma_s(G) \leq \Delta \cdot \gamma_p(G)$.

Proof: Let P be a PDS of G with $|P| = \gamma_p(G)$. Suppose that U is a set consisting of all vertices of $N[P]$ but excluding one neighbor of every vertex $u \in P$. Since $U \subset N[P]$, we have $|U| \leq \Delta \cdot |P|$. Clearly, $R_2(U) = R_2(N[P]) = V$. Thus, U is an SDS of G. This shows that $\gamma_s(G) \leq |U| \leq \Delta \cdot \gamma_p(G)$. ❑

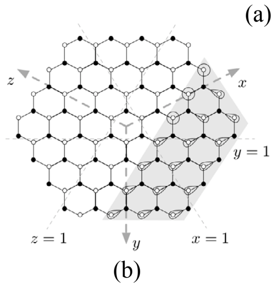
Lemma 2 Let S^* be a minimum *bias-SDS* in HM_t that contains V_2. Then, for any *SDS S* in HM_t, $|S \cap V_1| \geq |S^* \cap V_1|$.

Proof: Suppose to the contrary that there exists an SDS S in HM_t such that $|S \cap V_1| < |S^* \cap V_1|$. Let $S' = (S \cap V_1) \cup V_2$. Clearly, $S \subseteq S'$ and $|S' \cap V_1| = |S \cap V_1|$. Since S is an SDS in HM_t, this implies that S' is also an SDS in HM_t. Moreover, since S' contains V_2 as a subset, S' is a bias-SDS in HM_t. Thus, $|S' \cap V_1| < |S^* \cap V_1|$ contradicts the assumption that S^* is a minimum bias-SDS in HM_t. ❑

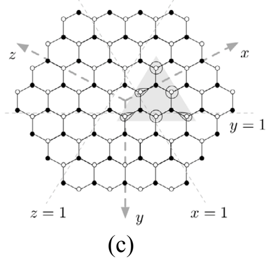
For HM_t, two disjoint sets $S, S' \subset V_1$ are said to be *extendable* if there exist two vertices $u \in R(S)$ and $v \in R(S')$ with $d(u, v) = 2$ such that $R(\{u, v\}) \neq \{u, v\}$. A set $S \subset V_1$ is called a *fit set* if either $|S| = 1$ or any partition of S, say $S = S' \cup S''$ with $S' \neq \emptyset$ and $S'' \neq \emptyset$, is extendable. Moreover, for a fit set S, a proper subset $S' \subset S$ is said to be a *maximal fit set within S* provided S' is a fit set and there is no other proper fit set $S'' \subset S$ such that $S'' \subset S'$. For a set $S \subseteq V_1$, we say that *S covers k x-values* if $|a \in \{1 - t, 2 - t, \ldots, t\} \mid (a, b, c) \in S\}| = k$. By a similar way, we can define the terms that *S covers k y-values* and *S covers k z-values*, respectively. For example, consider $S = \{(2, -1, 0), (3, -1, -1), (4, 0, -3)\} \in V_1$ in HM_4 (see Fig. 2 (a)), where each vertex of S is marked by a circle and each vertex of $R(S) \setminus S$ is marked by a shape of drip. If we partition S into two sets $S_1 = \{(2, -1, 0), (3, -1, -1)\}$ and $S_2 = \{(4, 0, -3)\}$, an easy observation shows that $d(u, v) > 2$ for any pair of vertices $u \in R(S_1)$ and $v \in R(S_2)$. Thus, S_1 and S_2 are not extendable. Note that $R(S_1)$ covers 2 x-values and $R(S_2)$ covers 5 x-values in this case. On the other hand, if we partition S into two sets $S_1 = \{(2, -1, 0)\}$ and $S_2 = \{(3, -1, -1), (4, 0, -3)\}$ and consider $u = (2, -1, 0)$ and $v = (3, -1, -1)$, then $d(u, v) = 2$ and $R(\{u, v\}) = \{(2, -1, 0), (3, -1, -1), (3, -2, 0)\} \neq \{u, v\}$. Thus, S_1 and S_2 are extendable in this case. As a result, S is not a fit set. Figs. 2 (b) and (c) show that $S = \{(2, 0, -1), (3, -1, -1), (4, -2, -1)\}$ and $S' = \{(1, 1, -1), (2, 0, -1), (2, -1, 0)\}$ are fit sets, respectively. In particular, $\{(2, 0, -1), (3, -1, -1)\}$ and $\{(3, -1, -1), (4, -2, -1)\}$ are maximal fit sets within S, but $\{(2, 0, -1), (4, -2, -1)\}$ is not.



(a)



(b)



(c)

Fig. 2. Examples of fit sets and unfit sets: (a) an unfit set; (b) and (c) two fit sets.

Lemma 3 Let S, $S' \subset V_1$ be two disjoint sets that are not extendable. Then, $R(S) \cup R(S') = R(S \cup S')$.

Proof: Clearly, $S \cup S' \subseteq R(S) \cup R(S') \subseteq R(S \cup S')$. We will show that if $w \in R(S \cup S') \setminus (S \cup S')$, then $w \in R(S) \cup R(S')$. Suppose to the contrary that there is a vertex $w \in R(S \cup S') \setminus (S \cup S')$ such that $w \notin R(S)$ and $w \notin R(S')$. In particular, we let w be a vertex observed by $S \cup S'$ that uses the minimum number of recursions of OR2. Since S and S' are not extendable, there do not exist two vertices $u \in R(S)$ and $v \in R(S')$ with $d(u, v) = 2$ such that $w \in R(\{u, v\})$. Since $w \in R(S \cup S') \setminus (S \cup S')$, either $w \in R(\{w_1, w_2\})$ or $w \in R(\{w_1\})$ for some vertices $w_1, w_2 \in V_1$. Here only the former case is considered, and the later case can be proved in a similar way. If $w \in R(\{w_1, w_2\})$, then w_1, w_2 and w have a common neighbor in V_2. Since $w \notin R(S)$ and $w \notin R(S')$, this implies $w_1, w_2 \notin R(S)$ and $w_1, w_2 \notin R(S')$. Thus, at least one of w_1 and w_2 must belong to $R(S \cup S') \setminus (R(S) \cup R(S'))$ and it can be observed by $S \cup S'$ using less number of recursions of OR2 than w. This leads to a contradiction. ❑

Lemma 4 Let S, $S' \subset V_1$ be two disjoint sets that are extendable. If S' is partitioned into S_1 and S_2 which are not extendable, then S and S_1 are extendable or S and S_2 are extendable.

Proof: Since S and $S' = S_1 \cup S_2$ are extendable, there exist vertices $u \in R(S)$ and $v \in R(S_1 \cup S_2)$ with $d(u, v) = 2$ such that $R(\{u, v\}) \neq \{u, v\}$. Since S_1 and S_2 are not extendable, by Lemma 3 we have $R(S_1 \cup S_2) = R(S_1) \cup R(S_2)$. Thus, $v \in R(S_1)$ or $v \in R(S_2)$, and the lemma follows. ❑

Lemma 5 Let S, $S' \subset V_1$ be two disjoint sets that are extendable. If both S and S' are fit sets, then so is $S \cup S'$.

Proof: To show that $S \cup S'$ is a fit set, we need to prove that any partition of $S \cup S'$ containing two nonempty subsets, say S_1 and S_2, are extendable. Let $T_1 = S_1 \cap S$, $T_2 = S_2 \cap S$, $T_1' = S_1 \cap S'$ and $T_2' = S_2 \cap S'$ (*i.e.*, $S = T_1 \cup T_2$, $S' = T_1' \cup T_2'$, $S_1 = T_1 \cup T_1'$ and $S_2 = T_2 \cup T_2'$). Since S and S' are extendable, if $T_1 = T_2' = \varnothing$ or $T_2 = T_1' = \varnothing$, then S_1 and S_2 are extendable. Thus, we consider $T_1 \neq \varnothing$ or $T_2' \neq \varnothing$ (respectively, $T_2 \neq \varnothing$ or $T_1' \neq \varnothing$). In addition, since both S_1 and S_2 are nonempty, we have $T_1 \neq \varnothing$ or $T_1' \neq \varnothing$ (respectively, $T_2 \neq \varnothing$ or $T_2' \neq \varnothing$). This implies that $T_1 \neq \varnothing$ and $T_2 \neq \varnothing$ or $T_1' \neq \varnothing$ and $T_2' \neq \varnothing$. Without loss of generality we consider the former case. Since $S = T_1 \cup T_2$ is a fit set, by definition, there exist vertices $u \in R(T_1)$ and $v \in R(T_2)$ with $d(u, v) = 2$ such that $R(\{u, v\}) \neq \{u, v\}$. Since $T_1 \subseteq S_1$ and $T_2 \subseteq S_2$, this further implies that S_1 and S_2 are extendable. ❑

Lemma 6 Let $S \subset V_1$ be a fit set and $U \subset S$ be a maximal fit set within S. Then, $S \setminus U$ is also a fit set.

Proof: Suppose to the contrary that $S \setminus U$ is not a fit set. Then, $S \setminus U$ can be partitioned into two nonempty subsets U_1 and U_2 that are not extendable. Since S is a fit set, U and $S \setminus U$ are extendable. By Lemma 4, U and U_1 are extendable or U and U_2 are extendable. Without loss of generality, suppose U and U_1 are extendable. If U_1 is fit, then so is $U \cup$

U_1 by Lemma 5. This contradicts that U is a maximal fit set within S. Thus, U_1 is not a fit set and there exists a partition of U_1 containing two nonempty sets U_1' and U_2' that are not extendable. Again, by Lemma 4, U and U_1' or U and U_2' are extendable. Without loss of generality, suppose U and U_1' are extendable. By the same argument, if we proceed the partition of U_1' repeatedly, there must exist a fit set containing at least one vertex that is extendable with U. Thus, this contradicts that U is a maximal fit set within S. ❑

Let $\ell_1, \ell_2, \ell_3 \in \{1-t, 2-t, \ldots, t\}$ be three integers. In the coordinate system suggested by Stojmenovic [23], a *positive triangle* with white vertices (*i.e.*, V_1) in HM_t is described by $\Delta(\ell_1, \ell_2, \ell_3) = \{(x, y, z) \in V_1 \mid x \leq \ell_1, y \leq \ell_2, z \leq \ell_3$ and $t \geq \max\{\ell_1+\ell_2, \ell_2+\ell_3, \ell_1+\ell_3\}\}$.

Note that the condition $\ell_i + \ell_j \leq t$ for each pair $i, j \in \{1, 2, 3\}$ with $i \neq j$ guarantees that the shape of $\Delta(\ell_1, \ell_2, \ell_3)$ forms a triangle in HM_t. For instance, the set of observed vertices in Fig. 2 (c) forms a positive triangle $\Delta_{(2,1,0)}$ in HM_4.

By contrast, a *negative triangle* with white vertices in HM_t is described by $\nabla(\ell_1, \ell_2, \ell_3) = \{(x, y, z) \in V_1 \mid x \geq \ell_1, y \geq \ell_2, z \geq \ell_3$ and $1-t \leq \min\{\ell_1+\ell_2, \ell_2+\ell_3, \ell_1+\ell_3\}\}$. In particular, if $\ell_1+\ell_2+\ell_3 = 1$, we let $\Delta(\ell_1, \ell_2, \ell_3) = \{(\ell_1, \ell_2, \ell_3)\}$ and $\nabla(\ell_1, \ell_2, \ell_3)$ undefined.

Similarly, a *positive x-trapezoid* with white vertices in HM_t is described by

$$\mathcal{T}_{x\leq\ell_1} = \{(x, y, z) \in V_1 \mid x \leq \ell_1 \leq 0, y \leq \ell_2 = t \text{ and } z \leq \ell_3 = t\}.$$

By contrast, a *negative x-trapezoid* with white vertices in HM_t is described by

$$\mathcal{T}_{x\geq\ell_1} = \{(x, y, z) \in V_1 \mid x \geq \ell_1 \geq 1, y \geq \ell_2 = 1-t \text{ and } z \geq \ell_3 = 1-t\}.$$

We can define positive (respectively, negative) y-trapezoid and z-trapezoid in a similar way. For instance, the set of observed vertices in Fig. 2 (b) forms a positive z-trapezoid $\mathcal{T}_{z\leq-1}$ in HM_4.

From the above definitions, it is easy to check the following propositions.

Proposition 7 Let $(x, y, z) \in V_1$ be a vertex in HM_t. Then,

$$R(\{(x,y,z)\}) = \begin{cases} \mathcal{T}_{x\leq 1-t} & \text{if } x = 1-t, \\ \mathcal{T}_{y\leq 1-t} & \text{if } y = 1-t, \\ \mathcal{T}_{z\leq 1-t} & \text{if } z = 1-t. \end{cases}$$

Proposition 7 means that if a vertex (x, y, z) with $x = 1-t$ (respectively, $y = 1-t$ or $z = 1-t$) has been observed, then so is every vertex with the same x-coordinate (respectively, y-coordinate or z-coordinate).

Proposition 8 Let $S \subset V_1$ be a set in HM_t and $X_k = \{(x, y, z) \in V_1 \mid x = k$ and $y + z = 1 - k\}$.

(1) For $1-t \leq k \leq t-1$, if $X_{k+1} \subseteq R(S)$, then $X_k \subseteq R(S)$.

(2) For $0 \le k \le t - 1$, if $X_k \subseteq R(S)$*, then* $X_{k+1} \subseteq R(S)$.

Proposition 8 means that if every vertex of X_{k+1} is observed, then so is every vertex of X_k; while the converse holds only for $0 \le k \le t - 1$. By symmetry, we have similar propositions for $Y_k = \{(x, y, z) \in V_1 \mid y = k \text{ and } x + z = 1 - k\}$ and $Z_k = \{(x, y, z) \in V_1 \mid z = k \text{ and } x + y = 1 - k\}$. The following two lemmas show that if a nonempty set $S \subset V_1$ together with V_2 is not a bias-SDS in HM_t, then $R(S)$ can determine certain shapes for those observed vertices.

Lemma 9 Let $S \subset V_1$ be a nonempty set in HM_t for $t \ge 2$. Then,

(1) $R(S)$ cannot form a negative x-, y- or z-*trapezoid*.
(2) If $R(S) = \mathcal{T}_{x\le\ell}$ (respectively, $R(S) = \mathcal{T}_{y\le\ell}$ or $R(S) = \mathcal{T}_{z\le\ell}$), then $\ell \le -1$.

***Proof*:** By symmetry, we only consider the proof for x-trapezoid. For statement (1), we suppose to the contrary that $R(S) = \mathcal{T}_{x\ge\ell}$ is a negative x-trapezoid for some $\ell \ge 1$. Clearly, $(t, 1 - t, 0) \in R(S)$. By Proposition 7, $R(\{(t, 1 - t, 0)\}) = \mathcal{T}_{y\le 1-t}$. Thus, $\mathcal{T}_{y\le 1-t} \subseteq R(S \cup \{(t, 1 - t, 0)\}) = R(S) = \mathcal{T}_{x\ge\ell}$. Since $(0, 1 - t, t) \in \mathcal{T}_{y\le 1-t}$, this implies $(0, 1 - t, t) \in \mathcal{T}_{x\ge\ell}$. Thus, $\mathcal{T}_{x\ge\ell}$ contains a vertex with x-coordinate less than ℓ, a contradiction.

For statement (2), we suppose to the contrary that $R(S) = \mathcal{T}_{x\le\ell}$ where $\ell \ge 0$. Clearly, $(0, 1 - t, t) \in R(S)$. By Proposition 7, an argument similar above shows that $(1, 1 - t, t - 1) \in \mathcal{T}_{y\le 1-t} = R(\{(0, 1 - t, t)\}) \subseteq R(S \cup \{(0, 1 - t, t)\}) = R(S) = \mathcal{T}_{x\le 0}$, a contradiction. ❑

Lemma 10 Let $S \subset V_1$ be a nonempty set in HM_t for $t \ge 2$. Then,

(1) $R(S)$ cannot form a negative triangle.
(2) If $R(S)$ is a positive triangle, then it cannot contain a vertex (x, y, z) with $x = 1 - t$, $y = 1 - t$ or $z = 1 - t$.

***Proof*:** For statement (1), we suppose to the contrary that $R(S) = \nabla(\ell_1, \ell_2, \ell_3)$ for some $\ell_1, \ell_2, \ell_3 \in \{1 - t, 2 - t, \ldots, t\}$ with $\min\{\ell_1 + \ell_2, \ell_2 + \ell_3, \ell_1 + \ell_3\} \ge 1 - t$. Since $t \ge 2$ and by the definition of a triangle, at most one of ℓ_1, ℓ_2 and ℓ_3 is equal to $1 - t$. Without loss of generality, we assume that $\ell_1 \ne 1 - t$ and $\ell_2 \ne 1 - t$. Clearly, $(\ell_1, \ell_2, 1 - (\ell_1 + \ell_2))$, $(\ell_1, \ell_2 + 1, -(\ell_1 + \ell_2)) \in \nabla(\ell_1, \ell_2, \ell_3)$ (see Fig. 3). Since $(\ell_1 - 1, \ell_2 + 1, 1 - (\ell_1 + \ell_2))$ is the only unob-

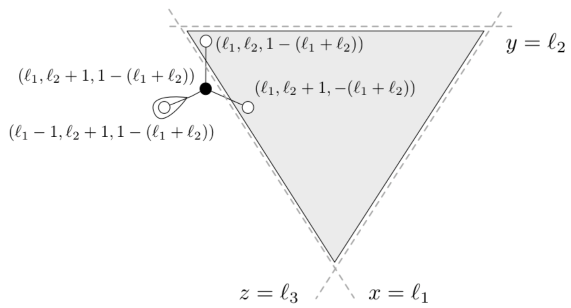


Fig. 3. Illustration of Lemma 10.

served neighbor of the vertex $(\ell_1, \ell_2 + 1, 1 - (\ell_1 + \ell_2)) \in V_2$, it becomes observed. Thus, $(\ell_1 - 1, \ell_2 + 1, 1 - (\ell_1 + \ell_2)) \in R(S)$. This contradicts that every vertex in $\nabla(\ell_1, \ell_2, \ell_3)$ has an x-coordinate at least ℓ_1.

For statement (2), without loss of generality, we suppose to the contrary that $R(S) = \Delta(\ell_1, \ell_2, \ell_3)$ for some $\ell_1, \ell_2, \ell_3 \in \{1 - t, 2 - t, \ldots, t\}$ with $\max\{\ell_1 + \ell_2, \ell_2 + \ell_3, \ell_1 + \ell_3\} \leq t$ and there is a vertex $(x = 1 - t, y, z) \in R(S)$. By Proposition 7, $R(S)$ contains all vertices (x, y, z) with $x = 1 - t$ (*i.e.*, $\mathcal{T}_{x\leq 1-t} \subseteq R(S)$). This contradicts that $R(S)$ is a positive triangle. ❑

The following proposition follows directly from the definitions of positive triangle and trapezoid.

Proposition 11 The following statements are true:

(1) Let $\ell \leq 0$ be an integer. The positive x-trapezoid $\mathcal{T}_{x\leq\ell}$ (respectively, y-trapezoid $\mathcal{T}_{y\leq\ell}$ and z-trapezoid $\mathcal{T}_{z\leq\ell}$) in HM_t covers $t + \ell$ x-values (respectively, y-values and z-values).
(2) Let $\ell_1, \ell_2, \ell_3 \in \{1 - t, 2 - t, \ldots, t\}$ be three integers with $\max\{\ell_1 + \ell_2, \ell_2 + \ell_3, \ell_1 + \ell_3\} \leq t$. The positive triangle $\Delta(\ell_1, \ell_2, \ell_3)$ in HM_t covers $\ell_1 + \ell_2 + \ell_3$ x-, y- and z-*values*.

Lemma 12 Let $S \subset V_1$ be a fit set in HM_t with $|S| < t$. Then, $R(S) \neq V_1$.

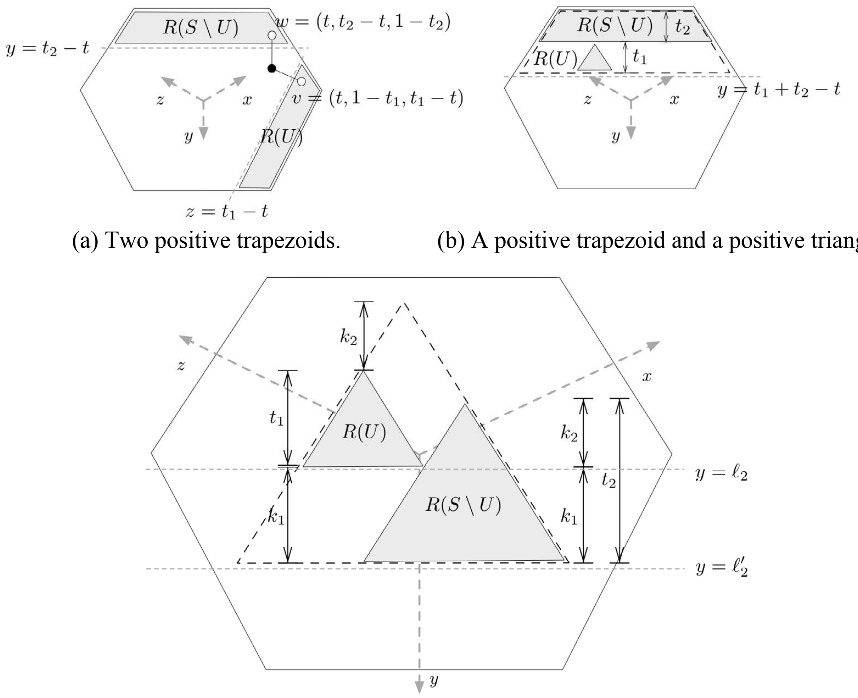
Proof: If $t = 1$, the lemma is clearly true. Let $t \geq 2$ be an integer. By Lemmas 9 and 10, we will show a stronger result that $R(S)$ is formed by either a positive x-trapezoid (respectively, y-trapezoid or z-trapezoid) covering at most $|S|$ x-values (respectively, y-values or z-values), or a positive triangle covering at most $|S|$ x-values. The proof is by induction on $|S|$. For $|S| = 1$, it is easy to prove that $R(S)$ is either a singleton (*i.e.*, a positive triangle) or a path in the boundary of HM_t (*i.e.*, a positive x-, y- or z-trapezoid).

For $|S| \geq 2$, we assume that the assertion holds for any proper subset of S, and let $U \subset S$ be a maximal fit set within S. By Lemma 6, $S \setminus U$ is also a fit set. Suppose $|U| = s_1$ and $|S \setminus U| = s_2$. From induction hypothesis, $R(U)$ and $R(S \setminus U)$ are positive x-, y- or z-trapezoids or positive triangles. Clearly, if $R(U) \subseteq R(S \setminus U)$ or $R(S \setminus U) \subseteq R(U)$, then $R(S)$ is still a positive x-, y- or z-trapezoid or positive triangle. For the case $R(U) \nsubseteq R(S \setminus U)$ and $R(S \setminus U) \nsubseteq R(U)$, we claim that both $R(U)$ and $R(S \setminus U)$ cannot be positive trapezoids simultaneously. Without loss of generality, suppose to the contrary that $R(U)$ is a positive z-trapezoid and $R(S \setminus U)$ is a positive y-trapezoid (see Fig. 4 (a)). Clearly, $R(U)$ covers at most s_1 z-values and $R(S \setminus U)$ covers at most s_2 y-values. By Proposition 11 we know that $R(U) = \mathcal{T}_{z\leq t_1-t}$ for some $t_1 \leq s_1$ and $R(S \setminus U) = \mathcal{T}_{y\leq t_2-t}$ for some $t_2 \leq s_2$. Since $t_1 + t_2 \leq s_1 + s_2 = |S| < t$, $R(U) \cap R(S \setminus U) = \varnothing$. In particular, $R(U)$ contains a vertex $v = (t, 1 - t_1, t_1 - t) \in V_1$ and $R(S \setminus U)$ contains a vertex $w = (t, t_2 - t, 1 - t_2) \in V_1$. Since S is fit, U and $S \setminus U$ are extendable. The only possible case for U and $S \setminus U$ being extendable must occur at v and w, *i.e.*, $d(v, w) = 2$ and $R(\{v, w\}) \neq \{v, w\}$. Thus, v and w have a common neighbor in V_2 (see the dark vertex in Fig. 4 (a)), and so $t_2 - t + 1 = 1 - t_1$. This contradicts that $t_1 + t_2 < t$.

In the following, without loss of generality, we only need to consider two cases.

Case 1: $R(U) = \Delta(\ell_1, \ell_2, \ell_3)$ is a positive triangle with $\ell_1 + \ell_2 + \ell_3 = t_1 \leq s_1$ and $R(S \setminus U) =$

$\mathcal{T}_{y \le t_2 - t}$ is a positive y-trapezoid with $t_2 \le s_2$ (see Fig. 4 (b)). Note that $R(U)$ and $R(S \setminus U)$ may have nonempty intersection. Clearly, $R(U)$ covers t_1 y-values and $R(S \setminus U)$ covers t_2 y-values, where $t_1 + t_2 < t$. Moreover, if $R(U) \nsubseteq R(S \setminus U)$, then $\ell_2 > t_2 - t$. Since U and $S \setminus U$ are extendable and both $R(U)$ and $R(S \setminus U)$ contain no vertex (x, y, z) with $y \ge 0$, by Proposition 8 every vertex of $R(S)$ must have a negative y-coordinate. In particular, $R(S) = \mathcal{T}_{y \le \ell}$ where $\ell = \max\{\ell_2, t_2 - t\} \le t_1 + t_2 - t$ (see the trapezoid bounded by dashed lines in Fig. 4 (b)).



(a) Two positive trapezoids. (b) A positive trapezoid and a positive triangle.

(c) Two positive triangles.

Fig. 4. Illustrations of Lemma 12.

Case 2: $R(U) = \Delta(\ell_1, \ell_2, \ell_3)$ with $\ell_1 + \ell_2 + \ell_3 = t_1 \le s_1$ and $R(S \setminus U) = \Delta(\ell'_1, \ell'_2, \ell'_3)$ with $\ell'_1 + \ell'_2 + \ell'_3 = t_2 \le s_2$ are two positive triangles (see Fig. 4 (c)). Note that $R(U)$ and $R(S \setminus U)$ may have nonempty intersection. For $R(U) \nsubseteq R(S \setminus U)$ and $R(S \setminus U) \nsubseteq R(U)$, without loss of generality we let $k_1 = \ell'_1 - \ell'_2 \ge 0$ and $k_2 = t_2 - k_1$. Since U and $S \setminus U$ are extendable, $R(S)$ must contain both $R(U)$ and $R(S \setminus U)$. Furthermore, it is not hard to see that $R(S)$ is formed by either a positive triangle or a positive trapezoid covering at most $t_1 + t_2$ y-values (see Fig. 4 (c) for the case that $R(S)$ is a triangle bounded by dashed lines that covers exactly $t_1 + (k_1 + k_2) = t_1 + t_2$ y-values). ❑

Lemma 13 Let S be a bias-*SDS* in HM_t that contains V_2 for $t \ge 2$. Then, $S \cap V_1$ is a fit set.

Proof: Let $U = S \cap V_1$. Since S is a bias-SDS in HM_t, $R(U) = V_1$. Suppose to the contrary that U is not a fit set. Then, U can be partitioned into two nonempty subsets U_1 and U_2 that are not extendable. By Lemma 3, we have $R(U_1) \cup R(U_2) = R(U) = V_1$. We first claim that if $u \in V_2$ is a vertex of degree three, then either $N(u) \subseteq R(U_1)$ or $N(u) \subseteq R(U_2)$. The assertion is clearly true since if $N(u) \cap R(U_1) \neq \varnothing$ and $N(u) \cap R(U_2) \neq \varnothing$, then U_1 and U_2 are extendable. Let $u = (0, 2-t, t)$ and without loss of generality we suppose $N(u) \subseteq R(U_1)$ (see Fig. 5 for HM_4). Since $(0, 2-t, t-1) \in R(U_1)$, this implies that all neighbors of $(1, 2-t, t-1)$ are contained in $R(U_1)$ (*i.e.*, $\{(0, 2-t, t-1), (1,1-t, t-1), (1, 2-t, t-2)\} \subset R(U_1)$). Furthermore, since $(1, 2-t, t-2) \in R(U_1)$, this implies that all neighbors of $(2, 2-t, t-2)$ are contained in $R(U_1)$. By the same argument, we obtain $Y_{1-t} \subset R(U_1)$ and $Y_{2-t} \subset R(U_1)$. Since every vertex (x, y, z) with $y = 3-t$ and $x + y + z = 2$ has a neighbor in Y_{2-t} and two neighbors in Y_{3-t}, we have $Y_{3-t} \subseteq R(U_1)$. Consequently, $R(U_1) = V_1$ by induction. Note that every vertex $v \in V_1$ must be adjacent to a vertex of degree three in V_2. Since $U_2 \neq \varnothing$ and $R(U_1) \cup R(U_2) = V_1$, we consider a vertex $v \in R(U_2) \subseteq V_1$ that is adjacent to a vertex $w \in V_2$ with $N(w) = \{v, v_1, v_2\} \subset V_1$. Clearly, $v_1 \in R(U_1)$. Since $d(v, v_1) = 2$ and $\{v, v_1, v_2\} \subseteq R(\{v, v_1\})$, this implies that U_1 and U_2 are extendable, a contradiction. ❑

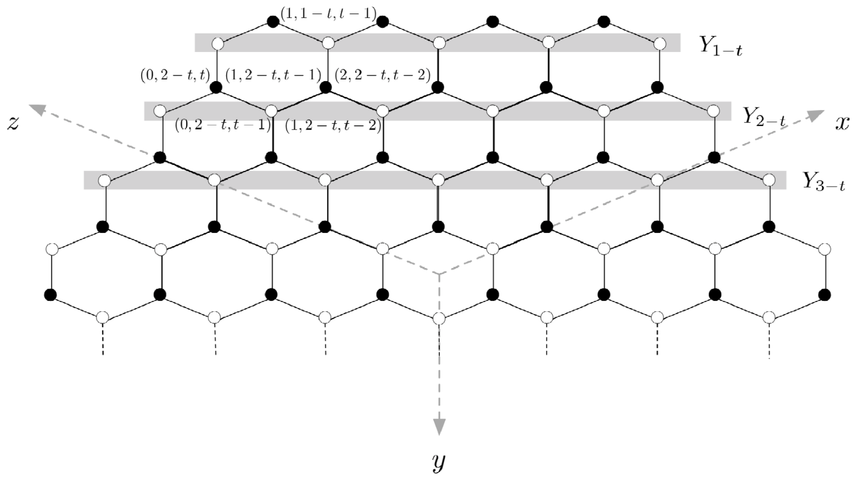


Fig. 5. Illustration of Lemma 13.

Lemma 14 $\gamma_p(HM_t) \geq \left\lceil \frac{2t}{3} \right\rceil$ for $t \geq 1$.

Proof: Let S^* be a minimum SDS in HM_t and S be a bias-SDS in HM_t that contains V_2 (*i.e.*, $R(S \cap V_1) = V_1$). By Lemma 13, $S \cap V_1$ is fit. Since we have already shown in Lemma 12 that $R(S \cap V_1) = V_1$ implies $|S \cap V_1| \geq t$ or $S \cap V_1$ is not a fit set, we conclude $|S \cap V_1| \geq t$ and every minimum bias-SDS in HM_t contining V_2 has at least t vertices of V_1. Lemma 2 further implies that every SDS in HM_t must contain at least t vertices of V_1. Thus, $|S^* \cap V_1| \geq t$. By symmetry, we have $|S^* \cap V_2| \geq t$. Hence, $\gamma_s(HM_t) = |S^*| \geq 2t$. By Lemma 1, $\gamma_s(HM_t) \leq 3 \cdot \gamma_p(HM_t)$. This completes the proof of the lemma. ❑

3. A SIMPLE ALGORITHM FOR FINDING A MINIMUM PDS IN HONEYCOMB MESHES

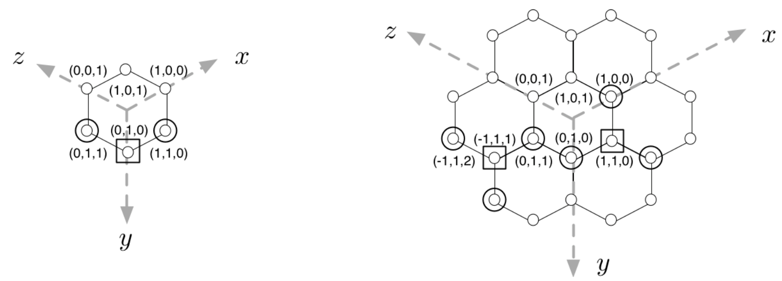
In what follows, we present an algorithm to find a minimum PDS in HM_t. Our algorithm is based on a simple rule and thus the time complexity is proportional to the size of such a PDS.

Algorithm PDS-ON-HM
Input: A honeycomb mesh HM_t.
Output: A PDS set P.

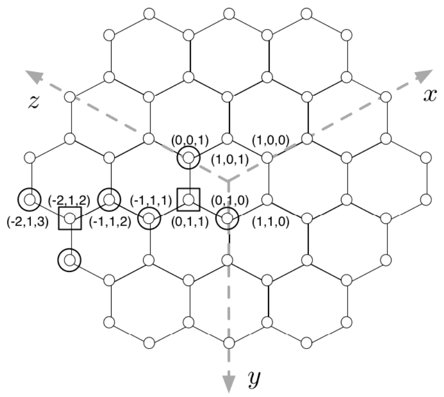
$P \leftarrow \emptyset$; $k \leftarrow t - 1 \bmod 3$;
if $k = 0$ **then** $(x, y, z) \leftarrow (0, 1, 0)$;
if $k = 1$ **then** $(x, y, z) \leftarrow (1, 1, 0)$;
if $k = 2$ **then** $(x, y, z) \leftarrow (0, 1, 1)$;
$P \leftarrow P \cup \{(x, y, z)\}$;
for $i = 1$ to $\lceil \frac{2t}{3} \rceil - 1$
    if $x + y + z = 1$ **then**
        $x \leftarrow x - 1$; $z \leftarrow z + 2$;
    else if $x + y + z = 2$ **then**
        $x \leftarrow x - 2$; $z \leftarrow z + 1$;
    $P \leftarrow P \cup \{(x, y, z)\}$;
endfor

Fig. 6. An algorithm for constructing a PDS in HM_t.

For example, Fig. 7 shows the power dominating sets generated by the algorithm PDS-ON-HM on HM_t for $t = 1, 2, 3$, where each vertex of P is marked by a square (*i.e.*, a PMU) and each vertex of $N(P)$ is marked by a circle. To show the correctness of the algorithm, we need the following terms. For $HM_t = (V, E)$, we redefine $X_k = \{(x, y, z) \in V \mid x = k \text{ and } 1 - k \leq y + z \leq 2 - k\}$. Also, define Y_k and Z_k by a similar way. In addition, let $V_x^+ = \bigcup_{k=1}^{t} X_k$ and $V_x^- = \bigcup_{k=1-t}^{0} X_k$. Again, by a similar way, we can define V_y^+, V_y^-, V_z^+ and V_z^-, respectively. Let $\prod_{xy}^+ = V_x^+ \cap V_y^+$ and $\prod_{xy}^- = V_x^- \cap V_y^-$. Also, define $\prod_{xz}^+$, $\prod_{xz}^-$, $\prod_{yz}^+$ and $\prod_{yz}^-$, similarly.



(a) A PDS $P = \{(0, 1, 0)\}$ on HM_1. (b) A PDS $P = \{(1, 1, 0), (-1, 1, 1)\}$ on HM_2.
Fig. 7. Examples of PDS generated by PDS-ON-HM.



(c) A PDS $P = \{(0, 1, 1), (2, 1, 2)\}$ on HM_3.

Fig. 7. (Cont'd) Examples of PDS generated by PDS-ON-HM.

Lemma 15 For HM_t with $t \geq 1$, if every vertex of V_x^+ (respectively, V_x^-, V_y^+, V_y^-, V_z^+ or V_z^-) is observed, then every vertex of V is observed.

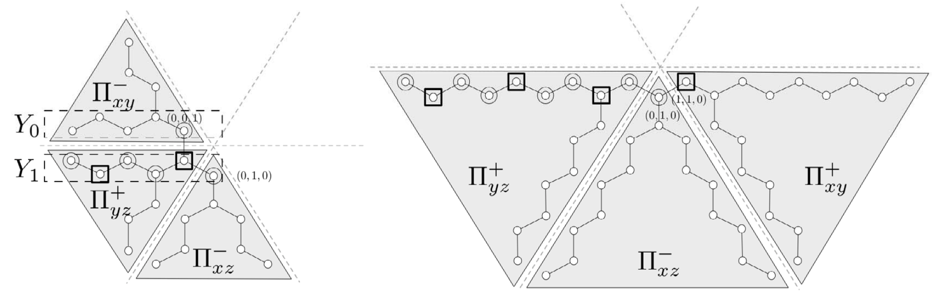
Proof: Suppose that all vertices of V_x^+ are observed. Since every vertex $(x, y, z) \in X_1$ with $y + z = 1$ has exactly one unobserved neighbor, this implies that every vertex $(x', y', z') \in X_0$ with $y' + z' = 1$ is observed. In particular, $(0, 1 - t, t)$ is observed and it has only one unobserved neighbor $(0, 2 - t, t)$. Thus, $(0, 2 - t, t)$ becomes observed. It follows immediately that every vertex $(x', y', z') \in X_0$ with $y' + z' = 2$ is observed. Therefore, we can show that all vertices of $X_0, X_{-1}, \ldots, X_{1-t}$ are observed by induction. ❑

Lemma 16 For HM_t with $t \geq 1$, Algorithm PDS-ON-HM correctly produces a PDS of size $\lceil \frac{2t}{3} \rceil$.

Proof: Let $k \equiv (t - 1) \bmod 3$ and let P be the set produced by PDS-ON-HM. Clearly, $|P| = \lceil \frac{2t}{3} \rceil$. In the following, we will show that $V_x^- \subseteq R_2(N[P])$ if $k = 2$, and $V_y^+ \subseteq R_2(N[P])$ otherwise. Thus, the correctness directly follows from Lemma 15.

Case 1: $k = 2$. Clearly, $V_x^- = \prod_{xy}^{-} \cup \prod_{yz}^{+} \cup \prod_{xz}^{-}$. We note that $\prod_{yz}^{+} \cap Y_1 \subset N[P]$. In particular, $(0, 0, 1) \in N(P)$ in $\prod_{xy}^{-}$ and $(0, 1, 0) \in N(P)$ in $\prod_{xz}^{-}$ (see Fig. 8 (a) for HM_3). Thus, every vertex of $(\prod_{yz}^{+} \cap Y_1) \cup \{(0, 0, 1), (0, 1, 0)\}$ is observed by OR1. It follows directly that every vertex of $\prod_{yz}^{+}$ is observed by OR2 recursively. Since $(0, 0, 1)$ and all vertices of $\prod_{yz}^{+} \cap Y_1$ are observed, every vertex of $\prod_{xy}^{-} \cap Y_0$ is observed by OR2, and this further implies that all vertices of $\prod_{xy}^{-}$ are observed by OR2 recursively. By the same argument, we can prove that all vertices in $\prod_{xz}^{-}$ are observed. As a consequence, all vertices of V_x^- are observed.

Case 2: $k \neq 2$. Clearly, $V_y^+ = \prod_{yz}^{+} \cup \prod_{xz}^{-} \cup \prod_{xy}^{+}$. We note that $\prod_{yz}^{+} \cap Y_1 \subset N[P]$. In particular, $(0, 1, 0) \in N(P)$ in $\prod_{xz}^{-}$ and $(1, 1, 0) \in N(P)$ in $\prod_{xy}^{+}$ (see Fig. 8 (b) for $k = 1$ in HM_5). Then, by an argument similar to Case 1, we can easily verify that all vertices of $\prod_{yz}^{+}$, $\prod_{xz}^{-}$ and $\prod_{xy}^{+}$, respectively, are observed by OR2 recursively. ❑



(a) An example for $k = 2$ in HM_3. (b) An example for $k = 1$ in HM_5.

Fig. 8. Illustrations of Lemma 16.

From Lemmas 14 and 16, we conclude the following.

Theorem 1 $\gamma_p(HM_t) = \lceil \frac{2t}{3} \rceil$ for $t \geq 1$.

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Kuo-Hua Kao (高國華) received the Ph.D. degree in Computer Science and Information Engineering from National Chi Nan University, Taiwan in 2010. His research interests include graph theory and algorithms.

Jou-Ming Chang (張肇明) received the B.S. degree in Applied Mathematics from Chinese Culture University, Taipei, Taiwan, in 1987, the M.S. degree in Information Management from National Chiao Tung University, Hsinchu, Taiwan, in 1992, and Ph.D. degree in Computer Science and Information Engineering from National Central University, Chungli, Taiwan, in 2001. He is working now as a Professor and the Chair at the Institute of Information and Decision Sciences in National Taipei College of Business (NTCB). His current research interests include algorithm design and analysis, graph theory parallel and distributed computing.

Yue-Li Wang (王有禮) received his B.S. and M.S. degrees both in Information Engineering Department of TamKang University in 1975 and 1979, respectively. In 1988, he received his Ph.D. degree in Information Engineering from National Tsing Hua University. Now, he is a Professor in the Department of Information Management of National Taiwan University of Science and Technology. His research interests include graph theory, algorithm analysis and parallel computing.

Shou-Hong Xu (徐碩鴻) received the B.S. degree in Information Management Department from Fu-Jen Catholic University, Taiwan in 2008, the M.S. degree in Computer Science and Information Engineering from National Chi-Nan University, Taiwan in 2010. He is an Engineer in the Data Processing Unit, Directorate General of Highway, Ministry of Transportation and Communication, Taiwan His research interests include algorithm design and analysis.

Justie Su-Tzu Juan (阮夙姿) received her B.S. degree in Applied Mathematics from Department of Mathematics, Fu Jen Catholic University in 1993, her M.S. and Ph.D. degrees in Applied Mathematics from National Chiao Tung University, Taiwan in 1996 and 2000, respectively. She is currently a Professor with the Department of Computer Science and Information Engineering, National Chi Nan University, Taiwan. Her research interests include graph theory, information security, cryptography, algorithms, and combinatorial mathematics.