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A New Tree Structure for Local Diagnosis

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Diagnosability is an important parameter to measure the fault tolerance of a multiprocessor system. If we only care about the state of a node, instead of doing the global diagnosis, Hsu and Tan proposed the idea of local diagnosis. Chiang and Tan provided an extended star structure to diagnose a node under comparison model. In this work, we evaluate the local diagnosability better by proposing a tree structure around this node. We provide the corresponding algorithm to diagnose the node. Simulation results are presented for different failure probability of a node in the tree and different percentage of faulty nodes in the tree, showing the performance of our algorithm.

Keywords: fault diagnosis, comparison model, diagnosis algorithm, local diagnosability, tree

1. INTRODUCTION

Fault diagnosis in multiprocessor systems has been widely studied. The procedure of identifying the faulty or fault-free status of each processor in a system is called system-level self diagnosis. Whenever processors are found faulty, they should be replaced with fault-free ones in order to guarantee that the system continues operating properly. The diagnosability of the system refers to the maximum number of faulty processors that can be identified.

To diagnose a multiprocessor system, several different models have been proposed [12]. One major method is called the comparison model, which was proposed by Maeng

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and Malek [9]. Under this model, the system performs diagnosis using a one-to-two testing: each processor sends two identical signals to each pair of its distinct neighbors and compares their responses. The result of the comparison depends on whether the two responses are agreed or not. Collecting all the comparison results, the system can decide the status of each node.

If we only want to know the state of a special node, Hsu and Tan [6] introduced the concept of local diagnosis. A local structure called an extended star was presented for guaranteeing a processor's local diagnosability. Occasionally, the local diagnosability guaranteed by the extended star is optimal. Usually, it is underestimated. In order to better evaluate the local diagnosability of a node, in this work we propose a tree sturcture and determine the local diagnosability of a node by the existence of this structure.

The rest of this paper is organized as follows: Section 2 provides preliminaries and background of system diagnosis. Section 3 introduces a new tree structure and the related algorithm to diagnose a node. In Section 4, we show the application of our new tree structure. Section 5 presents simulation results. In Section 6, we draw a conclusion.

2. PRELIMINARIES

For standard graph-theoretic terminology, we follow [1, 7]. In this paper, we use a finite and undirected graph $G(V,E)$ to represent a multiprocessor system where V is the vertex set of processors of the system and E is the edge set of communication links between two processors.

A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a subset of $V(G)$. We say that H is a subgraph of G induced by S if $V(H) = S$ and $E(H) = \{(u,v) \mid u,v \in V(S) \text{ and } (u,v) \in E(G)\}$. Let u be any vertex in G. The neighborhood of u in G, $N_G(u) = \{v \mid (u,v) \in E(G)\}$, is the set of vertices adjacent to u. The neighbor of a vertex subset A of a graph G is $N_G(A) = \cup_{u \in A} N_G(u)$, and the neighbor of A in subgraph H is $N_H(A) = N_G(A) \cap V(H)$. The degree of u in G, $\deg_G(u) = |N_G(u)|$, is the number of edges incident with u in G. We use $\delta(G) = \min_{v \in V(G)}\{d_G(v)\}$ to denote the minimum degree of the vertices of G.

Let A and B be any two sets. The difference set for A and B, $A - B$, is $\{x \mid x \in A$ and $x \notin B\}$, and the symmetric difference of A and B is $A \Delta B = (A - B) \cup (B - A)$.

A graph G is called t-diagnosable if the number of faulty vertices does not exceed t then all faulty vertices in G can be identified without replacement [12]. The diagnostic strategy of the comparison model are proposed as follows.

Comparison-based diagnosis [5, 10] requests a vertex to allocate the same system tasks to two distinct adjacent vertices and compare their responses. Let w, x and y be any three distinct vertices which $(w,x), (w,y) \in E(G)$. We use $\sigma_w(x,y)$ to represent the result of w compare the responses of x and y. Suppose that w is fault-free. If both x and y are fault-free, then $\sigma_w(x,y) = 0$; otherwise, $\sigma_w(x,y) = 1$. Suppose w is faulty. Then the test result is unreliable, that is, $\sigma_w(x,y) \in \{0,1\}$ no matter x and y are faulty or not. That is, if F is the set of faulty vertices then the outcome of a test result under comparison model is

$$\sigma_w(x,y) = \begin{cases} 0, & \text{if } \{w,x,y\} \cap F = \emptyset \\ 1, & \text{if } w \notin F \text{ and } \{x,y\} \cap F \neq \emptyset \\ 0 \text{ or } 1, & \text{if } w \in F \end{cases}$$

The necessary and sufficient conditions to identify if a pair of distinct faulty vertex subset is distinguishable or not as follows.

Theorem 2.1 *[13] For any two distinct vertex subsets F_1 and F_2 of a graph G, (F_1, F_2) is a distinguishable pair of G if and only if one of the following conditions is satisfied:*

(1) There are two vertices $w, u \in V(G) - (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \Delta F_2$ such that $(w, u) \in E(G)$ and $(w, v) \in E(G)$ (see Fig. 1(a) for an illustration);

(2) There are two vertices $u, v \in F_1 - F_2$ and there is a vertex $w \in V(G) - (F_1 \cup F_2)$ such that $(w, u) \in E(G)$ and $(w, v) \in E(G)$ (see Fig. 1(b) for an illustration);

(3) There are two vertices $u, v \in F_2 - F_1$ and there is a vertex $w \in V(G) - (F_1 \cup F_2)$ such that $(w, u) \in E(G)$ and $(w, v) \in E(G)$ (see Fig. 1(b) for an illustration).

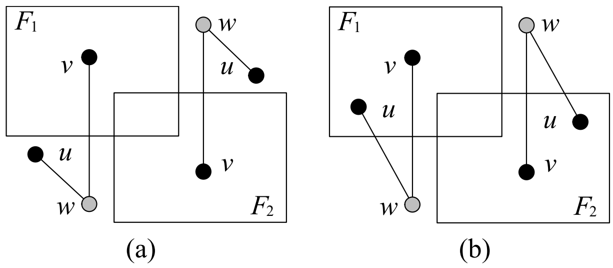

Fig. 1. Distinguishable pair under MM* model.

It follows from the definition of t-diagnosable and Theorem 2.1 that the following lemma holds.

Lemma 2.2 *A system G is t-diagnosable under MM* model if and only if, for each distinct pair F_1 and F_2 of subsets of $V(G)$ with $\max\{|F_1|, |F_2|\} \le t$, F_1 and F_2 are distinguishable.*

In contrast to the global sense in system diagnosis, Hsu and Tan present a local concept called the local diagnosability of a given node in a system. This method requires only the correct identification of the faulty or fault-free status of a single vertex. Below are two definitions that introduce the concept of local diagnosability.

Definition 2.3 *[4] A graph $G(V, E)$ is locally t-diagnosable at the vertex u, if given a test syndrome σ_F produced under the presence of a set of faulty vertices F containing the vertex u with $|F| \le t$, every set of faulty vertices F' consistent with σ_F and $|F'| \le t$, must also contain the vertex u.*

Definition 2.4 *[4] The local diagnosability $t_l(u)$ of a vertex u in a graph $G(V, E)$ is defined to be the maximum number of t for G being locally t-diagnosable at u, that is, $t_l(u) = \max\{t \mid G \text{ is locally } t\text{-diagnosable at } u\}$.*

The following result is another criteria for checking whether a vertex is locally t-diagnosable.

Lemma 2.5 *[6] A graph $G(V,E)$ is t-diagnosable at the vertex $u \in V$ if and only if for any two distinct sets of vertices $F_1, F_2 \subset V$, $|F_1| \leq t$, $F_2 \leq t$, $u \in F_1 \Delta F_2$, (F_1, F_2) is a distinguishable pair.*

The relationship between the local diagnosability and the traditional diagnosability is stated as follows.

Lemma 2.6 *[4] A graph $G(V,E)$ is t-diagnosable if and only if G is locally t-diagnosable at every vertex.*

Lemma 2.7 *[4] The diagnosability $t(G)$ of a graph $G(V,E)$ is equal to the minimum value among the local diagnosability of every vertex in G, that is, $t(G) = \min\{t_l(u) \mid u \in V(G)\}$.*

Under the comparison diagnosis model, an extended star structure for guaranteeing the local diagnosability of a given vertex is stated as below.

Definition 2.8 *[4] Let u be a vertex in a graph $G(V,E)$. An extended star $ES(u;n)$ of order n at the vertex u is defined as $ES(u;n) = (V(u;n), E(u;n))$, where the set of vertices $V(u;n) = \{u\} \cup \{v_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq 4\}$, the set of edges $E(u;n) = \{(u, v_{k,1}), (v_{k,1}, v_{k,2}), (v_{k,2}, v_{k,3}), (v_{k,3}, v_{k,4}) \mid 1 \leq k \leq n\}$ and $n \leq \deg_G(u)$. (See Fig. 2 for an illustration.)*

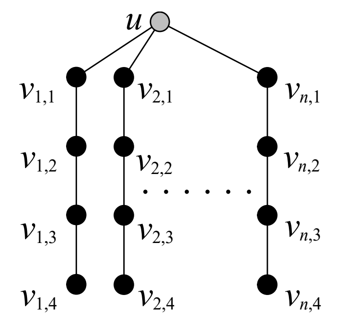

Fig. 2. Extended star structure $ES(u;n)$.

Chiang and Tan showed that n is a lower bound of $t_l(u)$ if there exists an extended star $ES(u;n)$ at u.

Theorem 2.9 *[4] Let u be a vertex in a graph $G(V,E)$. The local diagnosability of u is at least n if there exists an extended star $ES(u;n) \subseteq G$ at u.*

On the other hand, Hsu and Tan showed that the degree of u is a upper bound of $t_l(u)$.

Theorem 2.10 *[6] Let $G(V,E)$ be a graph and u be a vertex in the graph. Then local diagnosability of u is at most* $\deg_G(u)$.

The order n at a vertex is usually less than the degree of u. In order to better evaluate the local diagnosability of u, we propose a tree structure $T(u;a,b,c)$ around u in this paper.

3. A TREE STRUCTURE AND THE LOCAL DIAGNOSIS ALGORITHM

In this section, we first propose a tree structure to better evaluate the local diagnosability of a vertex. We provide the corresponding algorithm to diagnose a vertex based on the tree structure and the syndrome output by this structure.

Definition 3.1 *Let u be a vertex in a graph $G(V,E)$. For $a+b+c \leq \deg_G(u)$, a tree structure $T(u;a,b,c)$ of order $a+b+c$ at the vertex u is defined as $T(u;a,b,c) = (V(T(u;a,b,c)), E(T(u;a,b,c)))$, where the set of vertices $V(T(u;a,b,c)) = \{u\} \cup \{x_{i,j} \mid 1 \leq i \leq a, 1 \leq j \leq 4\} \cup \{y_{i,j} \mid 1 \leq i \leq b, 1 \leq j \leq 3\} \cup \{z_{i,j} \mid 1 \leq i \leq c, 1 \leq j \leq 2\}$, and the set of edges $E(T(u;a,b,c)) = \{(u,x_{k,1}),(x_{k,1},x_{k,2}),(x_{k,2},x_{k,3}),(x_{k,3},x_{k,4}) \mid 1 \leq k \leq a\} \cup \{(u,y_{k,1}),(y_{k,1},y_{k,2}),(y_{k,2},y_{k,3}) \mid 1 \leq k \leq b\} \cup \{(u,z_{k,1}),(z_{k,1},z_{k,2}) \mid 1 \leq k \leq c\}$. (See Fig. 3 for an illustration.)*

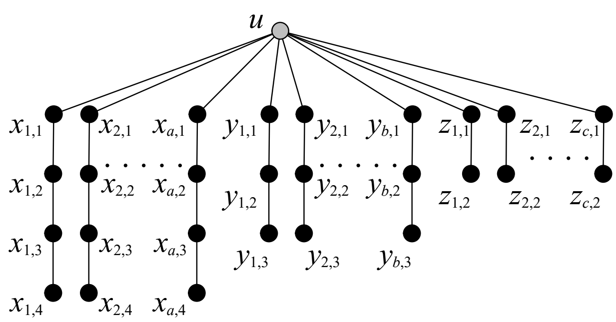

Fig. 3. A tree structure $T(u;a,b,c)$.

Theorem 3.2 *Let u be a vertex in a graph $G(V,E)$. The local diagnosability of u is at least $a+\lfloor \frac{b+c}{2} \rfloor$ if $b \leq c$ and there exists a tree structure $T(u;a,b,c) \subseteq G$ at u.*

Proof: Let t be a positive integer and $t \leq a+\lfloor \frac{b+c}{2} \rfloor$. We show that u is t-diagnosable. Let F_1, F_2 be two subsets of $V(G)$ such that $u \in F_1 \Delta F_2$ and $|F_1|,|F_2| \leq t$. Let $|F_1 \cap F_2| = p$, by assumption, we know that $0 \leq p \leq t-1$. Delete $F_1 \cap F_2$ from G, we consider the connected component u belongs to, denote it by C_u. After deleting $F_1 \cap F_2$ from G, among the $a+b+c$ branches around u there are at least $(a+b+c)-p$ complete branches in $T(u;a,b,c) - F_1 \cap F_2$. Notice that there are at most $2t-2p$ vertices in $F_1 \Delta F_2$.

There are $2a+b+c$ edges $\{(x_{i,1},x_{i,2}),(x_{i,3},x_{i,4}),(y_{j,1},y_{j,2}),(z_{k,1},z_{k,2}) \mid 1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq c\}$ independent to u in the structure $T(u;a,b,c)$. Next we show that after deleting $F_1 \cap F_2$ from $T(u;a,b,c)$ there is at least one edge independent to u left in C_u with both endpoints belong to $V(G)-(F_1\cup F_2)$. This edge can be connected to u by $F_1\Delta F_2$. Thus, Theorem 2.1 condition (1) holds. Therefore, F_1,F_2 is distinguishable.

We consider two cases.

Case 1. $p\ge a$. Among the $2a+b+c$ independent edges, at least $b+c-(p-a)$ edges left in C_u. Claim that at least one such independent edge belong to $G-(F_1\cup F_2)$. Otherwise, every independent edge has at least one endpoint belongs to $F_1\Delta F_2$ and $u\in F_1\Delta F_2$. So $|F_1\Delta F_2|\ge 1+(b+c)-(p-a)\ge 1+(t-p)+\lfloor\frac{b+c}{2}\rfloor=1+(t-p)+(t-a)>2t-2p$. It contradicts to the fact that $|F_1\Delta F_2|\le 2t-2p$.

Case 2. $p<a$. Among the $2a+b+c$ independent edges, at least $2(a-p)+b+c$ edges left in C_u. Among the $2(a-p)+b+c$ edges left in C_u, at least one edge belong to $G-(F_1\cup F_2)$. Otherwise, $|F_1\Delta F_2|\ge 1+2(a-p)+b+c\ge 1+2(t-p)>2t-2p$. It contradicts to the fact that $|F_1\Delta F_2|\le 2t-2p$.

Denote this edge by e, we know that e locates in one complete branch connected to u in $G-(F_1\cap F_2)$. We find Theorem 2.1 condition (1) structure in this branch. Hence, (F_1,F_2) is t-distinguishable. So the local diagnosability of u is at least $a+\lfloor\frac{b+c}{2}\rfloor$. □

Theorem 3.3 *Let u be a vertex in a graph $G(V,E)$. If $b>c$ and there exists a tree structure $T(u;a,b,c)\subseteq G$ at u then the local diagnosability of u is at least $a+\lfloor\frac{2b+c}{3}\rfloor$.*

Proof: Let t be a positive integer and $t\le a+\lfloor\frac{2b+c}{3}\rfloor$. We show that u is t-diagnosable. Let F_1,F_2 be two subsets of $V(G)$ such that $u\in F_1-F_2$ and $|F_1|,|F_2|\le t$. Let $|F_1\cap F_2|=p$, by assumption, we know that $0\le p\le t-1$. Delete $F_1\cap F_2$ from G, we consider the connected component u belongs to, denote it by C_u. After deleting $F_1\cap F_2$ from G, among the $a+b+c$ branches around u there are at least $(a+b+c)-p$ complete branches in $T(u;a,b,c)-F_1\cap F_2$. Notice that there are at most $2t-2p$ vertices in $F_1\Delta F_2$. By contrary, we assume that (F_1,F_2) is indistinguishable.

We know that $p\le t-1<a+b-1$, we consider two cases.

Case 1. $p<a$. Among the $(a-p)+b+c$ complete branches, at least $(a-p)$ branches has four edges; b branches has three edges; and c branches has two edges. For $i\in\{1,2,\dots,b\}$, we denote the vertices of three edges branch by $y_{i,1},y_{i,2},y_{i,3}$ such that $\{(u,y_{i,1}),(y_{i,1},y_{i,2}),(y_{i,2},y_{i,3})\}\subseteq E(G)$. Since (F_1,F_2) is indistinguishable, we have $|\{y_{i,1},y_{i,2},y_{i,3}\}\cap(F_1\Delta F_2)|\ge 1$. If $|\{y_{i,1},y_{i,2},y_{i,3}\}\cap(F_1\Delta F_2)|=1$ then we know $\{y_{i,1},y_{i,2},y_{i,3}\}\cap(F_1\Delta F_2)=\{y_{i,2}\}$ and $y_{i,2}\in F_2-F_1$ since (F_1,F_2) is indistinguishable. Suppose there are q such branches. That is, the other branches with three edges has at least two vertices belong to $F_1\Delta F_2$. Each complete branch with four edges (resp. two edges) has two edges $(x_{i,1},x_{i,2}),(x_{i,3},x_{i,4})$ (resp. one edge $(z_{i,1},z_{i,2})$) independent to u, at least one endpoint of each independent edge in C_u belong to $F_1\Delta F_2$ since (F_1,F_2) is indistinguishable. So $|F_1\Delta F_2|\ge 1+2(a-p)+c+2(b-q)+q=1+2(a-p)+2b+c-q$. We consider two subcases.

Subcase 1.1. $q\le\lfloor\frac{2b+c}{3}\rfloor$. Then $|F_1\Delta F_2|\ge 1+2(a-p)+2b+c-q\ge 1+2(a+\lfloor\frac{2b+c}{3}\rfloor-p)>2t-2p$. We get a contradiction.

Subcase 1.2. $q>\lfloor\frac{2b+c}{3}\rfloor$. Since $q\le|F_2-F_1|\le t-p\le a-p+\lfloor\frac{2b+c}{3}\rfloor$, let $q=\lfloor\frac{2b+c}{3}\rfloor+s$, then we have $1\le s\le a-p$.

Claim: Among the $a-p$ complete branches with four edges, at least s branches contains no vertices of F_2-F_1.

Otherwise, at least $a-p-s+1$ complete branches contains at least one vertex of F_2-F_1. Thus, we have $|F_2-F_1| \geq a-p-s+1+q = 1+a-p+\lfloor\frac{2b+c}{3}\rfloor > t-p$, a contradiction. Thus, our claim holds.

We consider the s complete branches which has four edges and contains no vertices of F_2-F_1, we know that $\{x_{i,1},x_{i,2},x_{i,3}\} \subseteq F_1-F_2$ since $u \in F_1-F_2$ and (F_1,F_2) is indistinguishable. Hence, $|F_1\Delta F_2| \geq 1+3s+2(a-p-s)+2(b-q)+q+c = 1+2(a-p)+2b+c-\lfloor\frac{2b+c}{3}\rfloor \geq 1+2(a+\lfloor\frac{2b+c}{3}\rfloor-p) > 2t-2p$. Again, we get a contradiction to $|F_1\Delta F_2| \leq 2t-2p$.

Case 2. $a \leq p < a+b-1$. Among the $a+b+c-p$ complete branches, at least $b-(p-a)$ branches has three edges and c branches has two edges. Similar to Case 1, we have $|F_1\Delta F_2| \geq 1+c+2(b-(p-a)-q)+q = 1+2(a-p)+2b+c-q$. We consider two subcases.

Subcase 2.1. $q \leq \lfloor\frac{2b+c}{3}\rfloor$. Then $|F_1\Delta F_2| \geq 1+2(a-p)+2b+c-q \geq 1+2(a+\lfloor\frac{2b+c}{3}\rfloor-p) > 2t-2p$. We get a contradiction.

Subcase 2.2. $q > \lfloor\frac{2b+c}{3}\rfloor$. Then $|F_2-F_1| \geq q > \lfloor\frac{2b+c}{3}\rfloor$. On the other hand, $|F_2-F_1| \leq t-p \leq \lfloor\frac{2b+c}{3}\rfloor$ since $t \leq a+\lfloor\frac{2b+c}{3}\rfloor$ and $p \geq a$. We get a condradiction. □

Denote the number $|\{i \mid (\sigma_{z_{i,1}}(u,z_{i,2})) = (j), 1 \leq i \leq c, 0 \leq j \leq 1\}|$ by c_j. Notice that $c_0+c_1 = c$. If u is faulty and $\sigma_{z_{i,1}}(u,z_{i,2}) = 0$, then $z_{i,1}$ is faulty. If u is faulty and $\sigma_{z_{i,1}}(u,z_{i,2}) = 1$, then the number of faulty vertices in the set $\{z_{i,1},z_{i,2}\}$ is uncertain. If u is fault-free and $\sigma_{z_{i,1}}(u,z_{i,2}) = 0$, then the number of faulty vertices in the set $\{z_{i,1},z_{i,2}\}$ is uncertain. If u is fault-free and $\sigma_{z_{i,1}}(u,z_{i,2}) = 1$, then at least one faulty vertex in the set $\{z_{i,1},z_{i,2}\}$. So we have the information shown in Table 1.

Table 1. The minimum number of faulty vertices in the set $\{z_{i,1},z_{i,2}\}$.

$(\sigma_{z_{i,1}}(u,z_{i,2}))$	$\min\lvert F\cap\{z_{i,1},z_{i,2}\}\rvert$	
	$u \in F$	$u \notin F$
(0)	1	0
(1)	0	1

We set $B_0 = \{(0,0)\}$, $B_1 = \{(1,0)\}$, $B_2 = \{(0,1),(1,1)\}$. Denote $|\{i \mid (\sigma_{y_{i,1}}(u,y_{i,2}), \sigma_{y_{i,2}}(y_{i,1},y_{i,3})) \in B_j, 1 \leq i \leq b\}|$ by b_j for $j \in \{0,1,2\}$. We know that $b_0+b_1+b_2 = b$. Suppose that u is faulty. If $(\sigma_{y_{i,1}}(u,y_{i,2}),\sigma_{y_{i,2}}(y_{i,1},y_{i,3})) = (0,0)$ then there are at least two faulty vertices in the set $\{y_{i,1},y_{i,2},y_{i,3}\}$ which are $y_{i,1}$ and $y_{i,2}$. Suppose that u is fault-free. If $(\sigma_{y_{i,1}}(u,y_{i,2}),\sigma_{y_{i,2}}(y_{i,1},y_{i,3})) = (0,0)$ then the number of faulty vertices in the set $\{y_{i,1},y_{i,2},y_{i,3}\}$ is uncertain. By similar analysis, we have the information in Table 2.

Table 2. The minimum number of faulty vertices in the set $\{y_{i,1},y_{i,2},y_{i,3}\}$.

$(\sigma_{y_{i,1}}(u,y_{i,2}),\sigma_{y_{i,2}}(y_{i,1},y_{i,3}))$	$\min\lvert F\cap\{y_{i,1},y_{i,2},y_{i,3}\}\rvert$	
	$u \in F$	$u \notin F$
$(0,0)$	2	0
$(1,0)$	0	1
$(0,1),(1,1)$	1	1

We set $A_0 = \{(0,0,0)\}$, $A_1 = \{(1,0,0)\}$, $A_2 = \{(i_1,i_2,i_3) \mid i_1,i_2,i_3 \in \{0,1\}\} - (A_0 \cup A_1)$. Denote the number $|\{i \mid (\sigma_{v_{i,1}}(u,w_{i,1}), \sigma_{v_{i,1}}(u,v_{i,2}), \sigma_{v_{i,2}}(v_{i,1},v_{i,3})) \in A_j, 1 \le i \le a\}|$ by a_j for $j \in \{0,1,2\}$. Notice that $a_0 + a_1 + a_2 = a$. The analysis of the number of faulty vertices in the set $\{x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}\}$, we refer to [4]. We have Table 3.

Table 3. The minimum number of faulty vertices in the set $\{x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}\}$.

$(\sigma_{x_{i,1}}(u,x_{i,2}), \sigma_{x_{i,2}}(x_{i,1},x_{i,3}), \sigma_{x_{i,3}}(x_{i,2},x_{i,4}))$	$\min\lvert F \cap \{x_{i,1},x_{i,2},x_{i,3},x_{i,4}\}\rvert$	
	$u \in F$	$u \notin F$
(0,0,0)	3	0
(1,0,0)	0	2
(0,0,1),(0,1,1)	2	1
(0,1,0), (1,0,1), (1,1,0), (1,1,1)	1	1

Based on the notation defined above, we can diagnose a vertex by the tree structure around it.

Algorithm 1: Local diagnosis algorithm for tree structure (**LDA**)

Input: A tree structure $T(u;a,b,c)$.
Output: The value is 0 or 1 if u is fault-free or faulty, respectively.

begin
  $a_0 \leftarrow |\{i \mid (\sigma_{x_{i,1}}(u,x_{i,2}), \sigma_{x_{i,2}}(x_{i,1},x_{i,3}), \sigma_{x_{i,3}}(x_{i,2},x_{i,4})) = (0,0,0), 1 \le i \le a\}|$;
  $a_1 \leftarrow |\{i \mid (\sigma_{x_{i,1}}(u,x_{i,2}), \sigma_{x_{i,2}}(x_{i,1},x_{i,3}), \sigma_{x_{i,3}}(x_{i,2},x_{i,4})) = (1,0,0), 1 \le i \le a\}|$;
  $a_2 = a - a_0 - a_1$;
  $b_0 \leftarrow |\{i \mid (\sigma_{y_{i,1}}(u,y_{i,2}), \sigma_{y_{i,2}}(y_{i,1},y_{i,3})) = (0,0), 1 \le i \le b\}|$;
  $b_1 \leftarrow |\{i \mid (\sigma_{y_{i,1}}(u,y_{i,2}), \sigma_{y_{i,2}}(y_{i,1},y_{i,3})) = (1,0), 1 \le i \le b\}|$;
  $b_2 = b - b_0 - b_1$;
  $c_0 \leftarrow |\{i \mid (\sigma_{z_{i,1}}(u,z_{i,2})) = (0), 1 \le i \le c\}|$;
  $c_1 = c - c_0$;
  if $b \le c$ **then**
    if $a_0 + \lfloor \frac{b_0+c_0}{2} \rfloor \ge a_1 + \lfloor \frac{b_1+b_2+c_1}{2} \rfloor$ **then return** 0;
    else return 1;
  end
  else
    if $2a_1 + a_2 + b_1 + b_2 + c_1 \le a + \lfloor \frac{2b+c}{3} \rfloor$ **then return** 0;
    else return 1;
  end
end

Theorem 3.4 *Let $T(u;a,b,c)$ be a tree of order $a+b+c$ at a vertex u, let F be any faulty set of G. If $b \le c$ and $|F| \le a + \lfloor \frac{b+c}{2} \rfloor$ then Algorithm 1 can identify the state of u correctly. That is, u is fault-free if $a_0 + \lfloor \frac{b_0+c_0}{2} \rfloor \ge a_1 + \lfloor \frac{b_1+b_2+c_1}{2} \rfloor$; otherwise, u is faulty.*

Proof: We prove this Theorem by contradiction. Suppose that u is faulty and $a_0+\lfloor\frac{b_0+c_0}{2}\rfloor \geq a_1+\lfloor\frac{b_1+b_2+c_1}{2}\rfloor$. According to Tables 1-3, we have $|F|\geq 1+3a_0+2b_0+c_0+a_2+b_2 \geq 1+a_0+\lfloor\frac{b_0+c_0}{2}\rfloor+a_0+\lceil\frac{b_0+c_0}{2}\rceil+a_2+b_2 \geq 1+a_1+\lfloor\frac{b_1+b_2+c_1}{2}\rfloor+a_0+\lceil\frac{b_0+c_0}{2}\rceil+a_2+b_2 > a+\lfloor\frac{b+c}{2}\rfloor$ which contradicts to $|F|\leq a+\lfloor\frac{b+c}{2}\rfloor$. Thus, u is fault-free if $a_0+\lfloor\frac{b_0+c_0}{2}\rfloor \geq a_1+\lfloor\frac{b_1+b_2+c_1}{2}\rfloor$.

Suppose that u is fault-free and $a_1+\lfloor\frac{b_1+b_2+c_1}{2}\rfloor > a_0+\lfloor\frac{b_0+c_0}{2}\rfloor$. According to Tables 1, 2 and 3, we have $|F|\geq 2a_1+a_2+b_1+b_2+c_1 \geq a_1+\lfloor\frac{b_1+b_2+c_1}{2}\rfloor+a_1+a_2+\lceil\frac{b_1+b_2+c_1}{2}\rceil \geq 1+a_0+\lfloor\frac{b_0+c_0}{2}\rfloor+a_1+a_2+\lceil\frac{b_1+b_2+c_1}{2}\rceil \geq 1+a+\lfloor\frac{b+c}{2}\rfloor$ which contradicts to $|F|\leq a+\lfloor\frac{b+c}{2}\rfloor$. Thus, u is faulty if $a_1+\lfloor\frac{b_1+b_2+c_1}{2}\rfloor > a_0+\lfloor\frac{b_0+c_0}{2}\rfloor$. □

Proposition 3.5 *If $b\leq c$ and there exists a tree structure $T(u;a,b,c)\subseteq G$ at u, then u is not locally $(1+a+\lfloor\frac{b+c}{2}\rfloor)$-diagnosable.*

Proof: Let $F_1\cap F_2=\{x_{k,1}\mid 1\leq k\leq a\}$, $F_1-F_2=\{u,z_{i,1}\mid 1\leq i\leq\lfloor\frac{b+c}{2}\rfloor\}$ and $F_2-F_1=\{y_{j,2},z_{i,1}\mid 1\leq j\leq b,\lfloor\frac{b+c}{2}\rfloor+1\leq i\leq c\}$. Then $|F_1|=1+a+\lfloor\frac{b+c}{2}\rfloor$, $|F_2|=a+b+(c-\lfloor\frac{b+c}{2}\rfloor)=a+\lceil\frac{b+c}{2}\rceil\leq 1+a+\lfloor\frac{b+c}{2}\rfloor$, $\max\{|F_1|,|F_2|\}=1+a+\lfloor\frac{b+c}{2}\rfloor$ and (F_1,F_2) is indistinguishable. (See Fig. 4 for an illustration.) □

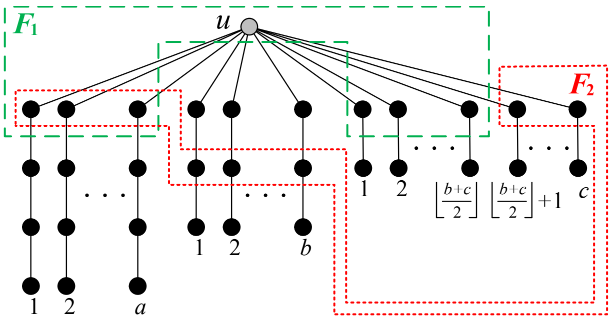

Fig. 4. An indistinguishable pair (F_1,F_2) with $\max\{|F_1|,|F_2|\}=1+a+\lfloor\frac{b+c}{2}\rfloor$.

If there exists a tree structure $T(u;a,b,c)$ at a vertex u and $b>c$ then we have the following theorem.

Theorem 3.6 *Let $T(u;a,b,c)$ be a tree of order $a+b+c$ at a vertex u, let F be any faulty set of G. If $b>c$ and $|F|\leq a+\lfloor\frac{2b+c}{3}\rfloor$ then Algorithm 1 can identify the state of u correctly. That is, u is fault-free if $2a_1+a_2+b_1+b_2+c_1\leq a+\lfloor\frac{2b+c}{3}\rfloor$; otherwise, u is faulty.*

Proof: We prove this Theorem by contradiction.

Suppose that u is faulty and $2a_1+a_2+b_1+b_2+c_1\leq a+\lfloor\frac{2b+c}{3}\rfloor$. According to Tables 1-3, we have $|F|\geq 1+3a_0+2b_0+c_0+a_2+b_2=1+3(a-a_1-a_2)+2(b-b_1-b_2)+(c-c_1)+a_2+b_2=1+3a+2b+c-3a_1-2a_2-2b_1-b_2-c_1\geq 1+3a+2b+c-4a_1-2a_2-2b_1-2b_2-2c_1\geq 1+3a+2b+c-2(a+\lfloor\frac{2b+c}{3}\rfloor)\geq 1+a+\lfloor\frac{2b+c}{3}\rfloor$

which contradicts to $|F| \le a + \lfloor \frac{2b+c}{3} \rfloor$. Thus, u is fault-free if $2a_1 + b_1 + c_1 + a_2 + b_2 \le a + \lfloor \frac{2b+c}{3} \rfloor$.

Suppose that u is fault-free and $2a_1 + a_2 + b_1 + b_2 + c_1 \ge a + \lfloor \frac{2b+c}{3} \rfloor + 1$. According to Tables 1, 2 and 3, we have $|F| \ge 2a_1 + a_2 + b_1 + b_2 + c_1 \ge 1 + a + \lfloor \frac{2b+c}{3} \rfloor$ which contradicts to $|F| \le a + \lfloor \frac{2b+c}{3} \rfloor$. Thus, u is faulty if $2a_1 + a_2 + b_1 + b_2 + c_1 \ge a + \lfloor \frac{2b+c}{3} \rfloor + 1$. □

Proposition 3.7 *If $b > c$ and there exists a tree structure $T(u;a,b,c)$ at u, then u is not locally $(1 + a + \lfloor \frac{2b+c}{3} \rfloor)$-diagnosable.*

Proof: Let $F_1 \cap F_2 = \{x_{k,1} \mid 1 \le k \le a\}$, $F_1 - F_2 = \{y_{i,2} \mid 1 \le i \le 1 + \lfloor \frac{2b+c}{3} \rfloor\}$ and $F_2 - F_1 = \{u, y_{i,1}, y_{i,2}, z_{j,1} \mid 2 + \lfloor \frac{2b+c}{3} \rfloor \le i \le b, 1 \le j \le c\}$. Then $|F_1| = 1 + a + \lfloor \frac{2b+c}{3} \rfloor$, $|F_2| = 1 + a + 2(b - \lfloor \frac{2b+c}{3} \rfloor - 1) + c = a + 2b + c - 2\lfloor \frac{2b+c}{3} \rfloor - 1 \le a + 3\lfloor \frac{2b+c}{3} \rfloor + 2 - 2\lfloor \frac{2b+c}{3} \rfloor - 1 = 1 + a + \lfloor \frac{2b+c}{3} \rfloor$, $\max\{|F_1|, |F_2|\} = 1 + a + \lfloor \frac{2b+c}{3} \rfloor$ and (F_1, F_2) is indistinguishable. (See Fig. 5 for an illustration.) □

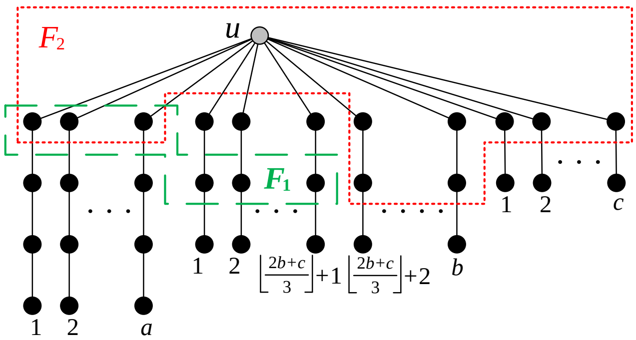

Fig. 5. An indistinguishable pair (F_1, F_2) with $\max\{|F_1|, |F_2|\} = 1 + a + \lfloor \frac{2b+c}{3} \rfloor$.

4. APPLICATION

The motivation of our paper is to improve the local diagnosability of a vertex. If the parameter n in the extended star $ES(u;n)$ is less than the degree of u then we try to make full use of the local structure around u. Next, we provide an example to show that by our new structure, the local diagnosability can be improved a lot for some vertices compared with the previous extended star.

Example 4.1 *Let $G = (V,E)$, where $V = \{u\} \cup \{x_i, y_i, z_i | 1 \le i \le 3k\}$ and $E = \{(u, x_i) | 1 \le i \le 3k\} \cup \{(x_i, y_i), (y_i, z_i) | 1 \le i \le 3k\}$.*

We know that $\deg_G(u) = 3k$. By the extended star $ES(u;n)$, we have $t_l(u) \ge 0$ since $n = 0$.

By the new structure we propose in this paper, we get that $t_l(u) \ge 2k$ since $a = c = 0$ and $b = 3k$.

As we can see, the local diagnosability of the vertex u has been improved by our new structure compared with the extended star.

In [2], the authors considered the local diagnosability of (n,k)-star graphs and Cayley graphs generated by 2-trees with some specified bounds of the number of missing edges by applying the extended star. In [3, 8], the authors considered the local diagnosability of star graphs S_n and Pancake graphs P_n with at most $n-3$ missing edges by showing the existence of extended stars. By our new tree structure, we still can measure the local diagnosability of a vertex even if the above mentioned graphs tolerate more edges.

5. SIMULATION RESULTS

In this section, we show the performance of the local diagnosis algorithm through the experimental data obtained in simulation. The purpose of simulation is to reveal how the different probabilities for a vertex to be faulty (*failure probability*) influence the accuracy of local diagnosis algorithm (Algorithm 1) with different values of a,b,c. On the other hand, we also simulate how different ratios of faulty vertices influence the accuracy of our local diagnosis algorithm.

We simulate 1000 times for each failure probability $p \in \{0.1,0.2,0.3,0.4,0.5,0.6\}$ and different values of a,b,c. The accuracy of u to be correctly diagnosed by Algorithm 1 is recorded in Table 4. As we can see from Table 4, the accuracy is 100% if $p=0.1$. If $p=0.2$ then we know that the accuracy is almost 100%. If $p=0.4$ then the accuracy is still above 80% except the case $b=c=0$. For the same failure probability, the accuracy increases slightly if the values of a,b,c increase.

Table 4. The influence of the probability for a vertex to be faulty on accuracy.

	$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$
$a=5,b=0,c=0$	100%	98%	91.2%	79.2%	65%	51.6%
$a=5,b=1,c=5$	100%	99.1%	93.9%	84.1%	77%	67.3%
$a=5,b=5,c=5$	100%	99.8%	95.9%	87.1%	77.1%	72.8%
$a=10,b=2,c=10$	100%	100%	98.1%	90%	80.1%	71.1%
$a=10,b=10,c=10$	100%	100%	98.6%	91.6%	80.2%	73.6%

We also simulate 1000 times for each case of ratio $r \in \{0.5,1,1.5,2,2.5,3\}$ and different values of a,b,c. The accuracy of u to be correctly diagnosed is recorded in Table 5. As we can see from Table 5, the accuracy is 100% if $r \le 1$. If $r=1.5$ then we know that the accuracy is at least 98%. If $r \le 2.5$ then the accuracy is still above 90% except the case $b=c=0$. For the same ratio, the accuracy increases slightly if the values of a,b,c increase.

As we can see from Tables 4 and 5 that if $b=c=0$ then accuracy is lower than $b>0,c>0$ for the same value of a ($a=5$) and the same value of p (resp. r). It means that the performance of our new tree structure $T(u;a,b,c)$ is much better than the extended star structure since $b=c=0$ in extended star $ES(u;a)$. Moreover, in both Tables 4 and 5, if the proportion of a,b,c is fixed, such as $a:b:c=5:1:5$ and $a:b:c=1:1:1$, then the accuracy increases little bit as the values of a,b,c increase. It is clear that for the same values of a and c, if the value of b increases then the accuracy increases for the same value of p (resp. r).

Table 5. The influence of the ratio of faulty vertices to the local diagnosability on accuracy.

	$r=0.5$	$r=1$	$r=1.5$	$r=2$	$r=2.5$	$r=3$
$a=5,b=0,c=0$	100%	100%	98.4%	89.7%	82.9%	77%
$a=5,b=1,c=5$	100%	100%	98.7%	94.2%	90.2%	86.1%
$a=5,b=5,c=5$	100%	100%	99.5%	96.1%	92.7%	88.7%
$a=10,b=2,c=10$	100%	100%	99.5%	96.5%	90.7%	88.8%
$a=10,b=10,c=10$	100%	100%	100%	97.6%	94.1%	89.1%

6. CONCLUDING REMARKS

In contrast to the traditional diagnosability, Hsu and Tan [6] introduced the concept of local diagnosability. Chiang and Tan [4] proposed an extended star to measure the local diagnosability of a node. Unfortunately, the local diagnosability guaranteed by the extended star is not always optimal. In order to better evaluate the local diagnosability, we provide a tree structure in this work. But our structure doesn't fit all networks and is not always optimal. There are other structures that can do better than the tree structure $T(u;a,b,c)$. In our future work, we will explore more structures to measure the local diagnosability and the g-good-neighbor conditional diagnosability proposed in [11] for multiprocessor systems.

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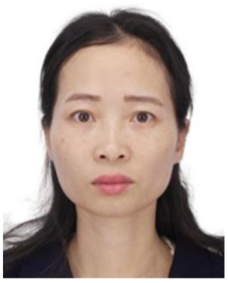

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