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Poincaré Section for Hide Coupled Dynamo Model

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Poincaré surface of section is an important tool in the dynamical system which allow us to imagine and understand the numerical solution behavior of the system. Hide's couple dynamo model is rich and worth to study. In this paper, we apply the Poincaré surface of section in three cases periodic orbit motion, regular motion and chaos motion.

Keywords: dynamo model, dynamical system, Poincaré map, stability, chaotic, regular motion

1. INTRODUCTION

Hide *et al*. in 1996 [1] extended the original Bullard system by introduced a motor. They have arisen the system of three non-dimensional nonlinear ordinary differential equation (ODEs) to Faraday disk dynamo model

$$\begin{aligned} \dot{x} &= x(y-1)-\beta z, \\ \dot{y} &= \alpha(1-x^2)-\kappa y, \\ \dot{z} &= x-\lambda z, \end{aligned} \tag{1}$$

where $\dot{x} = dx/d\tau$, *etc*. Eq. (1) govern the behavior dynamo system. τ is the independent variable and denoted by time t. Also, there are three dependent variables namely, $x(\tau), y(\tau)$ and $z(\tau)$ where $x(\tau)$ is rescaled electric current in the system, $y(\tau)$ denotes the angular rotation rate of the disk and $z(\tau)$ measures the angular speed of rotation of the motor. Moreover, we have four positive parameters appeared in Eq. (1) that control the dynamic of the system; α and β measure the applied couple and the inverse moment of the inertia of the armature, respectively; κ and λ measure the mechanical friction in the disk and motor respectively. Self-exciting dynamos are nonlinear electromechanical system and one of the elementary models of self-excitation of a magnetic field by moving conductor. Such a models has an own importance which can be used to clarify the dynamo action that is believed as analogous to the generation of Earth, Sun and cosmic bodies magnetic field.

In this paper, we use an important tool in the dynamical system to imagine and understand the behavior in Hide's coupled dynamo model with three different cases. These

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cases are most common motions and show how the motion behaves once we vary the parameters.

2. PROBLEM AND ITS HISTORY

Geophysicists approve that the most magnetic fields seen naturally in many places (*e.g.* in the Earth and other planets, in Sun and other stars and Galaxies that are produced by the self exciting magnetohydrodynamic (MHD) dynamo in the electrically conducting fluid that can be found in their interiors). Larmor [2] is the first scientist who noticed the valuable of the mechanism of self exciting dynamo. In 1919, he suggested that the Sun's magnetic (field) is generated by (dynamo) motional induction including the movement of electrically conducting fluid that can be found in the convection regions of the inside solar [3-6]. Dynamo theory target is to understand such complicated issues that depend on MHD with highly nonlinear partial differential equations. Due to difficulty to investigate such issue, the need to find suitable 'low-dimensional' analogues which governed by nonlinear differential equations is becomes necessary. The first like analogues was suggested by Bullard in 1955 [7] who analysed the behavior of a single disk dynamo and found that the system exhibits periodic relaxation oscillation but does not support aperiodic reversals. The system was

$$\begin{aligned} \dot{x} &= x(y-1), \\ \dot{y} &= \alpha(1-x^2) - \kappa y. \end{aligned} \tag{2}$$

Rikitaki in 1958 [8] subsequently coupled two such Bollard dynamo system together which caused chaotic reversal of the current, unlike the system produced by Bullard and the system has written in non-dimension form [9] as follows

$$\begin{aligned} \dot{x} &= zy - \mu x, \\ \dot{y} &= x(z-\gamma) - \mu y, \\ \dot{z} &= 1 - xy. \end{aligned}$$

Other extensions which introduced resistive shunt have also produced chaotic behavior [10]; see also Ghil and Childress 1987 [4].

Once the three parameters β, λ and κ are equal to zero, then we will have the originally important study (2) which investigated by Bullard in 1955 [7]. Many dynamical systems can be derive from the above systems. See for example Allahem in [11].

3. PRELIMINARY RESULT

The reflection symmetry occurs in the system (1) which are remain invariant and unchanged under the transformation if

$$x \to -x, y \to y, z \to -z \tag{3}$$

Hide and others [1, 12, 13] have studied the stability of the equilibrium point in their paper. They started with determination of the steady equilibrium of the system in Eq. (1)

by assuming $\dot{x} = \dot{y} = \dot{z} = 0$. Thus, the steady state is

$$(x,y,z) = (0, \frac{\alpha}{\kappa}, 0). \tag{4}$$

For the second steady states, we get

$$x = \pm\sqrt{1 - \frac{\kappa}{\alpha}(1 + \frac{\beta}{\lambda})},$$

and then we obtain

$$z = \frac{x}{\lambda}.$$

Thus, we find

$$(x,y,z) = (\pm\sqrt{1 - \frac{\kappa}{\alpha}(1 + \frac{\beta}{\lambda})},\ \frac{\beta}{\lambda} + 1,\ \frac{x}{\lambda}). \tag{5}$$

3.1 Linear Stability of the $(0, \alpha/\kappa, 0)$ and its Bifurcation

In order to study the stability of the solution in Eqs. (4) and (5), we need to apply the linearization method. Then the Jacobi matrix which ends with eigenvalues as follows:

$$\sigma_1 = -\kappa,\ \sigma_{2,3} = \frac{1}{2}\{\ \frac{\alpha}{\kappa} - 1 - \lambda \pm \sqrt{(\frac{\alpha}{\kappa} - 1 + \lambda)^2 - 4\beta}\ \}$$

Two cases are arisen in the studying the stability of Eq. (4): Firstly, if all the three eigenvalues have a real part with negative signs, then the solution is stable. The another one is a local bifurcation, which is the change in stability and this happen if at least one of real part of eigenvalue changes its sign. The last situation occurs if one or more eigenvalues are equal to zero, which implies that the system subject to a steady bifurcation or if we have a complex conjugate pair of eigenvalues, which implies that the system subject to a Hopf bifurcation. Additionally, $\alpha, \beta, \kappa,$ and λ parameter space are divided by the planes of bifurcation point into different regions of behavior.

Bifurcation of $(0, \alpha/\kappa, 0)$:
The zero eigenvalue occurs if $c = 0$ in cubic eigenvalues equation

$$\sigma^3 + a\sigma^2 + b\sigma + c = 0,$$

where

$$a = \kappa - \frac{\alpha}{\kappa} + 1 + \lambda,$$
$$b = \lambda\kappa - \frac{\alpha\lambda}{\kappa} + \lambda - \alpha + k + \beta,$$
$$c = \beta\kappa - \alpha\lambda + \kappa\lambda.$$

Thus, we have

$$\beta\kappa - \alpha\lambda + \kappa\lambda = 0, \qquad \frac{\alpha}{\kappa} = \frac{\beta}{\lambda} + 1.$$

Hence, pitchfork bifurcations occur along the line

$$\frac{\alpha}{\kappa} = \frac{\beta}{\lambda} + 1,$$

and is called a line of symmetry-breaking bifurcation. At this point a couple of solution in Eq. (5) branch from the steady solution in Eq. (4). As λ is decreased ($\lambda \to 0$) the slope of the symmetry-breaking line is increased in the $(\beta, \alpha/\kappa)$ plane and when $\lambda = 0$ the line coincident with abscissa, where $\beta = 0$.

Furthermore, when

$$\frac{\alpha}{\kappa} = 1 + \lambda \ \text{ and } \ \lambda^2 < \beta$$

a complex conjugate pair of eigenvalues exists and hence a line of Hopf bifurcation occurs which is shown in line labelled as h_1 in Fig. 5 in [14].

3.2 Linear Stability of the $\left(\sqrt{1 - \frac{\kappa}{\alpha}\left(1 + \frac{\beta}{\lambda}\right)}, 1 + \frac{\beta}{\lambda}, \frac{x}{\lambda}\right)$

A linear stability study of (1) for the second equilibrium point (5) presents the Jacobian matrix that ends with cubic eigenvalues equation and can be written as

$$\sigma^3 + a\sigma^2 + b\sigma + c = 0$$

where

$$a = \kappa + \lambda - \frac{\beta}{\lambda}$$
$$b = 2(\alpha - \kappa) + \kappa\lambda - 3\frac{\beta\kappa}{\lambda}$$
$$c = 2[\alpha\lambda - \kappa\lambda + \kappa\beta]$$

Similar to the analysis for solution in Eq. (4), we have a single zero eigenvalue if $\alpha/\kappa = 1 + \beta/\lambda$ also we have a double zero eigenvalues exactly at the point P in [14]. Moreover, the Hopf bifurcation in Eq. (5) occurs if

$$ab = c \quad \text{and} \quad b > 0.$$

3.3 Numerical Integration

Numerical integrations play an important role in showing complicated behavior of the system. Instead of introduce a complete description of the $(\beta - \alpha/\kappa)$ space, we investigate the change in the behavior in selected parameters. To integrate the equation of the system (1), we apply the most famous numerical method namely, fourth-order Runge-Kutta method with appropriate time stepping. The numerical solutions are integrated for fininte time to make sure that end state solutions are stable.

Now some results for a fixed parameter values will be presented. Starting with the case $\lambda = 0$ which is physically unrealistic, however studying this case is advantageous as

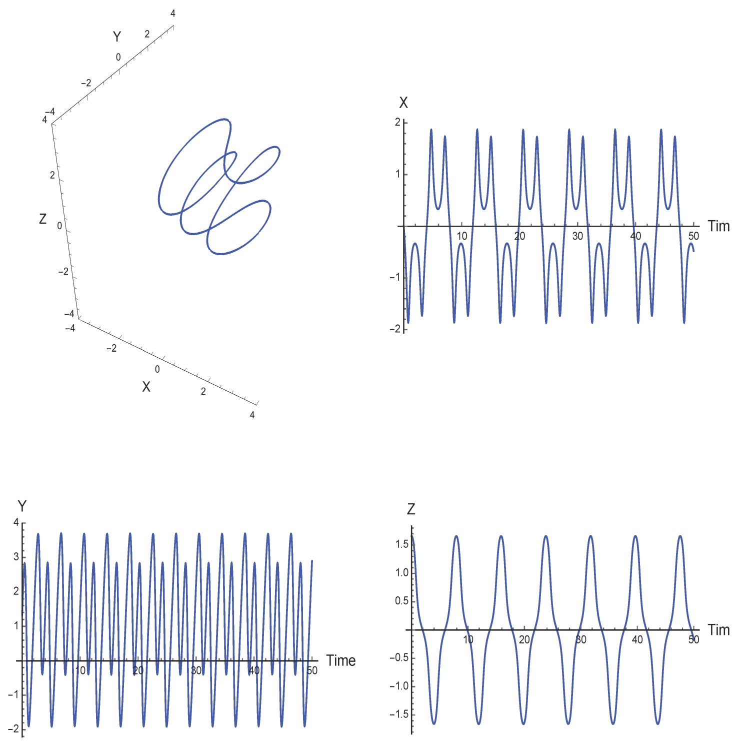

Fig. 1. Numerical integrations, $\lambda = 0.0, \kappa = 0.1, \beta = 1.0$ and $\alpha = 5.0$ a phase space plot, the axes are x,y and z; the time of the x-coordinate and the time of the y and z-coordinates, respectively.

a starting point. From the bifurcation diagram in [14], it is clearly that as $\lambda \to 0$ the slop of the line $s$in the $(\beta, \alpha/\kappa)$ plane becomes bigger until this lines *i.e.*, symmetry-breaking bifurcation coincident with the axis.

The obtained results of numerical integration are presented as following. In Fig. 1, the numerical integration all the time approaches to the solution presented by (1) for all the values of parameters α, β, κ and λ. However, once α has value more than 0.1 and also $\kappa = 0.1$, periodic solution is discovered. Note that as α increase the periodic solutions arise more and more. furthermore, in the case $\lambda \neq 0$ which is physically realistic, chaotic solutions are presented. In the Fig. 1 where $\alpha = 0.15$. It is clearly that the periodic orbit is occurred as shown in the Fig. 1 and lies nearly in the $y = \alpha/\kappa = 1$ plane which located near to bifurcation point as expected via the analyzing study. Fig. 1 displays four representations of the solution for the parameters value $\kappa = 0.1$, $\lambda = 0.0$, $\alpha = 0.15$, and $\beta = 1.0$ where all the numerical solutions are provided by using the same format. In more details, Fig. 1 provides the phase space plot. For the purpose of understanding this plotting, Fig. 1 also shows the projection of the phase space in the $x-y-z$. In addition, plots in Fig. 1 represent the time-series of the x and y-coordinates and the time-series of the z-coordinate, respectively. Here, The nature of this specified parameters values is seen clearly in the projection on the $x-y-z$ as well as in the time series of y coordinate. It is clearly that the solution winds once rounds the line $x = 1$,$y = 1$ and once round the line

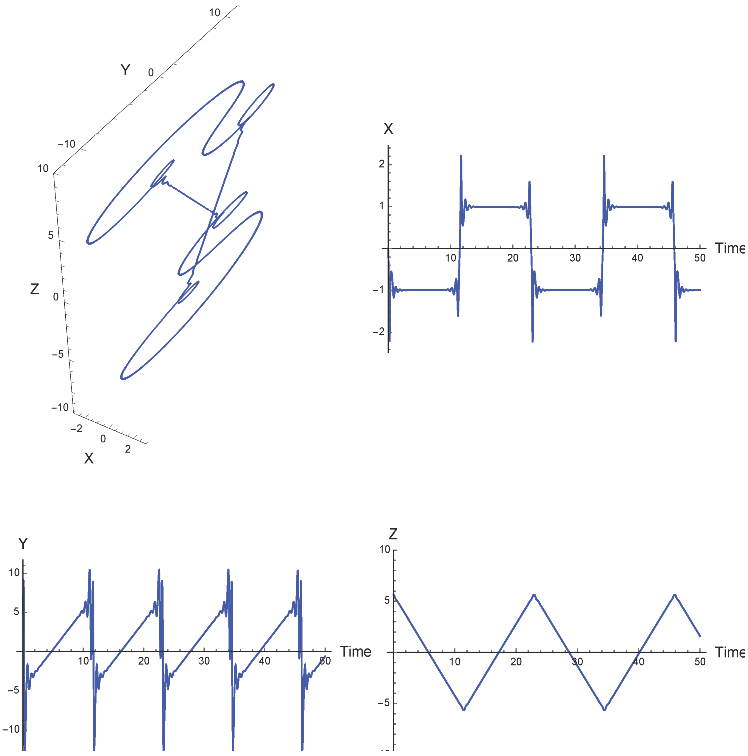

Fig. 2. Numerical integrations, $\lambda = 0.0, \kappa = 0.1, \beta = 1.0$ and $\alpha = 50$ a phase space plot, the axes are x,y and z; the time of the x-coordinate and the time of the y and z-coordinates, respectively.

$x = -1$, $y = 1$. A physical interpretation of the system is given as the current oscillates about zero in comparing with the oscillations in the speed of the motor, whereas the speed of rotation of the disk stays at an approximately constant level.

As we know in the case α increase the periodic orbit expand in the size. Hence the several of solutions states are possible where founding these solution depends on the initial conditions. An example of that

Fig. 2 which can be described as possess two distinct 'humps' in the time series x (the current) as shown in Fig. 2 while in the time series y the disk periodically exchanges from rotating in one direction to rotating in the opposite direction as shown in Fig. 2. Here the solution winds around each line twice in comparison with the solution in Fig. 2. The nearby initial condition provide the regular motion which will study and clarify in the next section.

Chaotic solutions are observed by increasing α to 100 and choose $\lambda = 1.0$. These solutions go and back between the two unstable periodic cycles which represent chaotic attractor similar to the well-known Loranz attractor [15]. Fig. 3 shows the numerical solution behavior for existence chosen parameters. It shows rapidly flipping and oscillating chaotic behavior which worth to investigate and understand using one of the most tool in dynamical system namely, Poincaré surface of section.

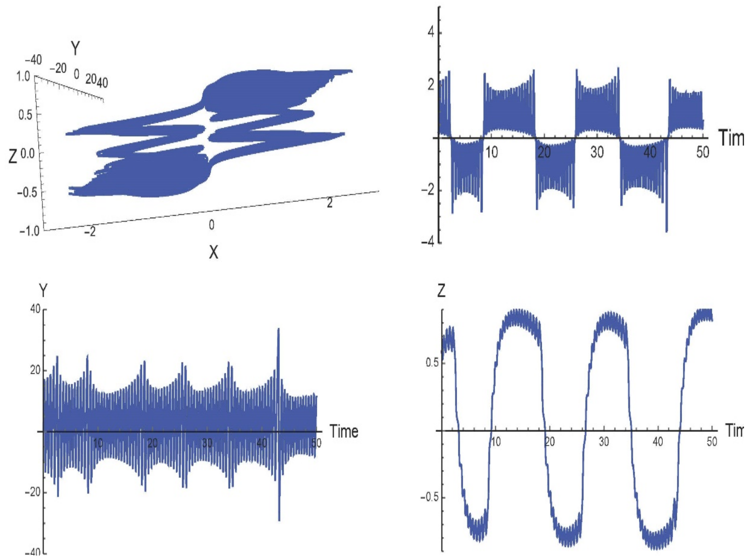

Fig. 3. Numerical integrations, $\lambda = 1.0, \kappa = 0.1, \beta = 1.01$ and $\alpha = 100$ a phase space plot, the axes are x,y and z; the time of the x-coordinate and the time of the y and z-coordinates, respectively.

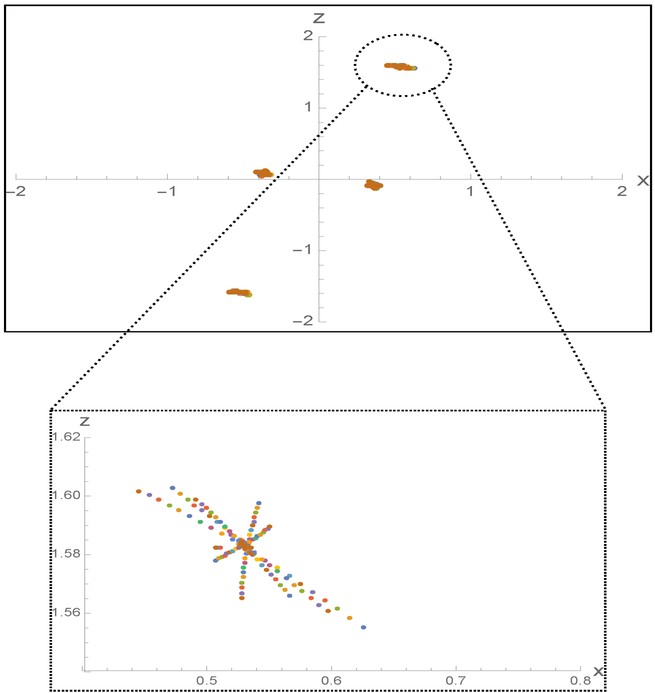

Fig. 4. Poincaré surface of section $y = 0.0$ with parameters $\lambda = 0.0, \kappa = 0.1, \beta = 1.0$ and $\alpha = 5.0$.

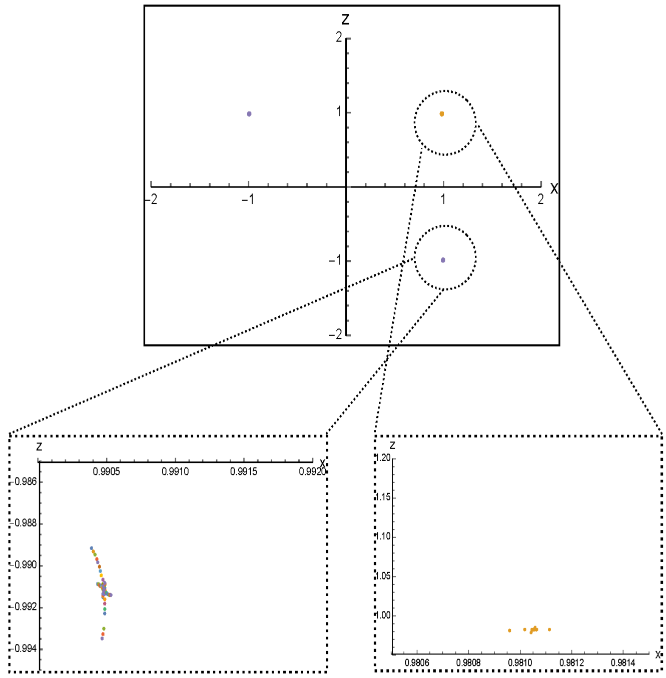

Fig. 5. Poincaré surface of section $y = 0.0$ with parameters $\lambda = 0.0, \kappa = 0.1, \beta = 1.0$ and $\alpha = 50$.

4. MAIN RESULT

The dynamics of Hide's coupled dynamo model flow is understood clarify by observing the dynamics behavior of the flow on a particular section in the phase space. Chosen such a section is very important and one of the famous section in the dynamical system is called the Poincaré section. It helps to imagine the underlying dynamics [16]. This section is set in the middle of the numerical projection of the phase space in the $x-y-z$. The intersections of the flow with the section provide a discrete mapping known as the Poincaré map or the return map [17-19]. In fact, Poincaré surface of section [17-19] is a helpful tool to study qualitative properties of a dynamical system behavior . More precisely, It presents the asymptotic stability of periodic or almost periodic orbits and the regular motion as well as the chaotic motion as collection of points. Basically, the Poincaré map [17-19] describes how such points on the Poincaré section get mapped back onto itself by the flow and intersected again. Three cases has been studied including periodic motion, regular motion and chaotic motion. For all cases the Poincaré surface of section set to be $y = 0.0$. First of all, Fig. 4 shows the periodic motion where the periodic orbit intersects the Poincaré surface of section in four point to return back to the first intersection. Fig. 4 does not show the periodic solution intersection but show also many periodic motion in the neighbourhood of the periodic orbit which appear as cross lines in the bottom of Fig. 4. These cross lines provide a hint of the stable and unstable manifold attached to the periodic orbit. In the second case, the Poincaré surface of section

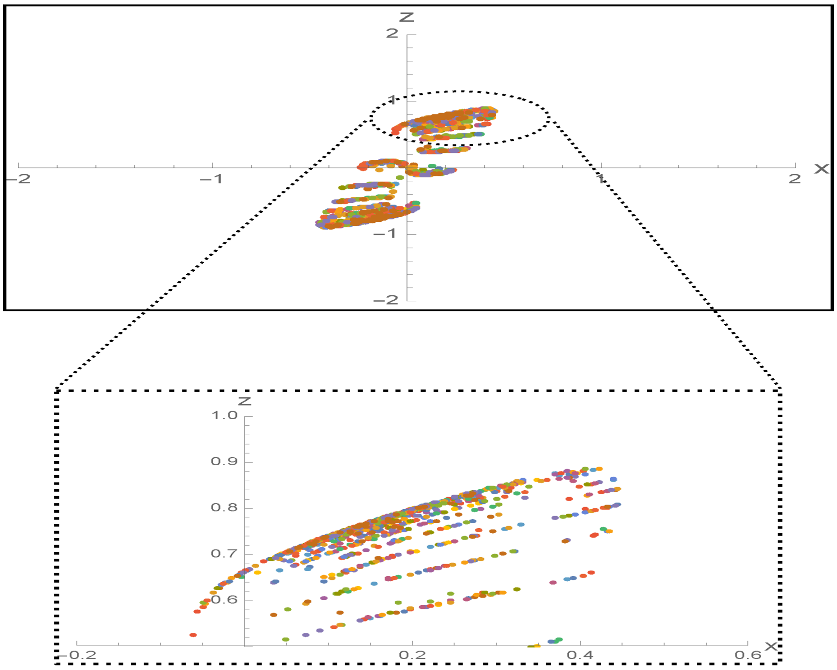

Fig. 6. Poincaré surface of section $y = 0.0$ with parameters $\lambda = 1.0, \kappa = 0.1, \beta = 1.01$ and $\alpha = 100$.

shows quasipreiodic orbit which over loop and intersect the Poincaré surface of section many time to return to the initial condition. The different between the first and second case, Poincaré surface of section has been intersected four time before return to the initial condition while the second case needs many time intersections to get back. The last case, the parameters choice provide the chaotic motion. Among this chaos appearance, we set the Poincaré surface of section also to be $y = 0.0$. Fig. 6 shows random points which refer to the intersection points without specific order. This kind of Poincaré surface of section is obvious in chaotic motion. In the bottom of Fig. 6, we attempt to zoom in existence area to make the figure much readable.

5. CONCLUSION

The Poincaré surface of section is a rare phenomenon as it can be obtained by a very special choice of Poincaré section. Actually, the Poincaré section [8-15] is help us to read and understand some dynamical systems. We have seen how to obtain two dimensional Poincaré surface of section for Hide's dynamo model which is three dimensional dynamical system. Three cases has been studied with three different motions including periodic motion, regular motion and chaotic motion. A further cases can be studied by setting the parameters in different value to show the beauty of the dynamical system theory. Finally, Poincaré surface of section tool with good choice can be used in many famous dynamical system to read, understand and visualize the solution behavior .

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