JOURNAL OF INFORMATION SCIENCE AND ENGINEERING 36, 1223-1232 (2020)
DOI: 10.6688/JISE.202011_36(6).0006

Comparison Between Milstein and Exact Coupling Methods using MATLAB for a Particular Two-Dimensional Stochastic Differential Equation*

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We compare Milstein and exact coupling methods for the strong approximation of solutions to stochastic differential equations (SDE), which are driven by Brownian motion. Both of these methods attain an order one convergence under the nondegeneracy assumption of the diffusion term for the exact coupling method. We also compare their implementation using MATLAB. A particular two-dimensional SDE is used in the implementation for comparing their results. Moreover, the performance of both methods and the amount of time required to obtain the result are also analyzed. It is interesting to mention that this comparison is very important in several areas, such as stochastic analysis, financial mathematics and some physical applications.

Keywords: Stochastic differential equations (SDE), numerical solution of stochastic differential equations, Milstein method for solving DSE, exact coupling, coupling

1. INTRODUCTION

It is observed in the literature that the researches to obtain solutions of stochastic differential equations (SDEs) are progressing rapidly and attracting the interest of many researchers working in this field. Recently, numerical solutions to stochastic differential equations have become popular with the development of computing simulations. The solution of SDEs has potential applications in many fields, such as economics, finance and physics [1, 2]. Some studies have been done to find the strong solutions of the stochastic differential equation to obtain approximations of order greater than $\frac{1}{2}$. In [1, 3, 4] anthors developed new methods and used the truncation of the Fourier series of the Wiener process to approximate the double integrals in higher dimension. However, these methods required significant computational time. In [5], Fournier used the quadratic Wasserstein metric approach to approximate the Euler scheme. In [6], Davie described the application of the Wasserstein bound to the solutions of the stochastic differential equation and used

Received January 6, 2020; revised February 6 & March 12, 2020; accepted March 31, 2020.
Communicated by Mahmoud Abdel-Aty.
* The author gratefully acknowledge Qassim University, represented by the deanship of scientific research, on the material support for this research under the number (3952-cos-2018-1-14-S) during the academic year 2018.

a version of Komlós, Major, and Tusnády method to obtained order one approximation under some assumptions. Yang, Chen and Wan [7] used the Itô-Taylor expansion with a specific condition to approximate the densities of multivariate. Under some conditions, Alfonsi, Jourdain, and Kohatsu-Higa [9, 10] developed the Wasserstein convergence for the Euler method and proved an $O(h^{(\frac{2}{3}-\varepsilon)})$ convergence for a one-dimensional diffusion process. Gaines and Lyons [11] developed a new method based on rectangle-wedge-tail for two-dimensional SDEs. A new method for the two dimensional SDEs using the condition on the endpoints was presented in [12]. The bound of an approximation to a pathwise solution on a given probability space was established by Gyöngy and Krylov in [13] using the coupling method. Some simulation methods for the stochastic differential equation have been studied in [14]. The MATLAB implementation for the Euler and Milstein methods in one- and two-dimensional SDEs was introduced in [15]. For the interested reader to know more about the accomplishment of simulation of stochastic differential equation, we refer to [16, 17, 18]. In this paper, two numerical methods named Milstein and exact coupling are used and compared. This is based on the standard order one Milstein scheme using a Wasserstein matrix with the condition that SDE has invertible diffusion. We show the MATLAB implementation for both methods and compare the result as well as the computational time. We used *(MATLAB ver. R2017b)* software to obtain the implementation and approximation results. The rest of this article is organized as follows. In Section 2, certain results concerning SDEs are reviewed, and the Davie method [8] is studied. In the last section, the comparison between Milstein and exact coupling methods is presented, and a numerical implementation is provided to demonstrate the convergence behavior for 2-dimensional SDEs using invertible diffusion.

2. SCHEMES OF APPROXIMATION OF SDEs

A standard Brownian motion, or standard Wiener process, over an interval $[0,T]$ is a random variable $W(t)$, which depends continuously on a time $t \in [0,T]$, if the following conditions are satisfied:

(i) $W(0) = 0$ (with probability one).

(ii) For $0 \leq s < t \leq T$, the random variable given by the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$. Equivalently $W(t) - W(s)$ is $N(0, t - s)$.

(iii) For $0 \leq s < t < u < v \leq T$, the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

2.1 Definition

Let $\{W(t)\}_{t\geq 0}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$, where $a = a(t,x)$ be a d-dimensional vector function(called *drift* coefficient) and $b = b(t,x)$ is a $d \times d$-matrix function(called *diffusion* coefficient). The stochastic process $X = X(t)$, which is considered in this work, can be described by *stochastic differential equations*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \quad t \in [0, T]. \tag{1}$$

Let the initial condition $X(0) = x$ be an $\mathscr{F}_0$-measurable random vector in $\mathbb{R}^d$. A $\mathscr{F}_t$-adapted stochastic process $X = (X(t))_{t\geq 0}$ is called a solution of Eq. (1), if

$$X(t) = X(0) + \int_0^t a(s,X(s))ds + \int_0^t b(s,X(s))dW(s), \tag{2}$$

holds a, s. The conditions that the integral processes

$$\int_0^t a(s,X(s))ds, \qquad \int_0^t b(s,X(s))dW(s),$$

are well-defined and required for Eq. (2) to hold. For the functions $a(s,X(s))$ and $b(s, X(s))$, we have the following conditions:

$$E\int_0^t b^2(s,X(s))ds < \infty, \tag{3}$$

and almost surely for all $t \geq 0$,

$$\int_0^t |a(s,X(s))|ds < \infty. \tag{4}$$

One property for the stochastic integral is

$$\int_0^t W(s)dW(s) = \frac{1}{2}\int_0^t d(W^2(s)) - \frac{1}{2}\int_0^t ds = \frac{1}{2}W^2(t) - \frac{t}{2}.$$

For more details on the stochastic integral, the interested reader is referred to [1].

2.2 Existence and Uniqueness Theorems

Consider the following conditions:

(i) **Measurability:** Let $a:[0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ and $b:[0,\infty)\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$ be jointly Borel measurable in $[t_0,T]\times\mathbb{R}^d$.

(ii) **Lipschitz condition:** There is a constant $A > 0$ such that $|a(t,x)-a(t,y)| \leq A|x-y|$, and $|b(t,x)-b(t,y)| \leq A|x-y|$, for all $t \in [t_0,T]$ and $x,y \in \mathbb{R}$.

(iii) **Growth condition:** There is a constant $K > 0$ such that $|a(t,x)|^2 \leq K^2(1+|x|^2)$, and $|b(t,x)|^2 \leq K^2(1+|x|^2)$, for all $t \in [t_0,T]$ and $x,y \in \mathbb{R}$.

The following theorem gives sufficient conditions for the existence and uniqueness of a solution of a stochastic differential equation.

2.3 Theorem

Under the previous conditions (i)-(iii), the stochastic differential Eq. (1) has a unique solution $X(t) \in [t_0,T]$ with

$$\sup_{t_0\leq t\leq T} E(|X(t)|^2) < \infty.$$

Proof. See Theorem 4.5.3 [1]. □

2.3.1 Approximation schemes

In this subsection, we briefly review the schemes of the Euler-Maruyama, Milstein and Davie methods. Consider an Itô stochastic differential equation

$$dX_i(t) = a_i(t,X(t))dt + \sum_{k=1}^{d} b_{ik}(t,X(t))dW_k(t),\ X_i(0) = X_i^{(0)}, \tag{5}$$

on an interval $[0,T]$, where i =1,...,d , for a d-dimensional vector $X(t)$, with a d-dimensional driving Brownian path $W(t)$. If the coefficients $b_{ik}(t,X(t))$ satisfy a global Lipschitz condition $|a(t,x)-a(t,y)| \leq A|x-y|$, and $|b(t,x)-b(t,y)| \leq A|x-y|$, for all $t \in [t_0,T]$ with $x,y \in \mathbb{R}$ and $A > 0$ is a constant. If a_i and b_i are continuous in t, for each X, then the Eq. (5) has a unique solution $X(t)$. This is a process adapted to the filtration induced by the Brownian motion. Under these conditions, the solutions satisfies $E(|X(t)|^p) < \infty$, for each $p \in [1,\infty]$ and $t \in [0,T]$. The standard method to the pathwise approximation of the solution of Eq. (5), is to divide $[0,T]$ into a finite number N of equal intervals with length equal to $h = T/N$. The simplest form of such approximation for the SDE by using only the linear term in the Taylor expansion, gives the following Euler-Maruyama scheme

$$x_i^{(j+1)} = x_i^{(j)} + \sum_{k=1}^{d} b_{ik}(x^{(j)})\Delta W_k^{(j)}, \tag{6}$$

where $\Delta W_k^{(j)} = W_k((j+1)h) - W_k(jh)$. Now, we represent a scheme which is proposed by Milstein and gives an order one strong Taylor scheme.

$$\begin{aligned} x_i^{(j+1)} &= x_i^{(j)} + a_i(jh,x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh,x^{(j)})\Delta W_k^{(j)} \\ &\quad + \sum_{k,l=1}^{d} \rho_{ikl}(jh,x^{(j)})M_{kl}^{(j)}, \end{aligned} \tag{7}$$

where $\Delta W_k^{(j)} = W_k((j+1)h) - W_k(jh)$,

$M_{kl}^{(j)} = \int_{jh}^{(j+1)h} \{W_k(t) - W_k(jh)\}dW_l(t)$, and $\rho_{ikl}(t,x) = \sum_{m=1}^{q} b_{mk}(t,x)\frac{\partial b_{il}}{\partial x_m}(t,x)$. If the following condition

$$\rho_{ikl}(t,x) = \rho_{ilk}(t,x), \tag{8}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$ and all i,k,l holds, then the Milstein scheme reduces to

$$\begin{aligned} x_i^{(j+1)} &= x_i^{(j)} + a_i(jh,x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh,x^{(j)})\Delta W_k^{(j)} \\ &\quad + \sum_{k,l=1}^{d} \rho_{ikl}(jh,x^{(j)})B_{kl}^{(j)}. \end{aligned} \tag{9}$$

This depends only on the generation of the Brownian motion $\Delta W_k^{(j)}$. It can be implemented with a special class of equations for the Milstein method using only the Brownian

motion $\Delta W_k^{(j)}$. This comes from the observation that

$$M_{kl}^{(j)} + A_{lk}^{(j)} = 2B_{kl}^{(j)} \text{ where } B_{kl}^{(j)} = \tfrac{1}{2}\Delta W_k^{(j)}\Delta W_l^{(j)}, \text{ for } k \neq l$$
$$\text{and } B_{kk}^{(j)} = \tfrac{1}{2}\{(\Delta W_k^{(j)})^2 - h\}.$$

Scheme (9) achieves an order of 1, for $d = 1$. However, for the dimension $d > 1$, we obtain the order $\frac{1}{2}$. According to Davie's exact coupling method, we could modify the previous scheme (9). This gives order 1 under invertible diffusion conditions. One can implement the Milstein method by generating the random variables $\Delta W_k^{(j)}$ and $M_{kl}^{(j)}$ separately and then combines them to obtain the RHS of the scheme (9). According to Davie's method, we attempt to generate the following:

$$Y := \sum b_{ik}(jh, x^{(j)})\Delta W_k^{(j)} + \sum \rho_{ikl}(jh, x^{(j)})M_{kl}^{(j)},$$

directly. If we have a scheme

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum b_{ik}(jh, x^{(j)})X_k^{(j)} + \sum \rho_{ikl}(jh, x^{(j)})(X_k^{(j)}X_l^{(j)} - h\delta_{kl}), \tag{10}$$

with the increment $X_k^{(j)}$ being independent and $N(0,h)$ being random variables, then it is same as scheme (9) by replacing $\Delta W_k^{(j)}$ with $X_k^{(j)}$ and we do not assume $\Delta W_k^{(j)} = X_k^{(j)}$.

2.4 Strong Order of Convergence

A discrete time approximation x_h with the step-size h converges strongly with order γ at time $T = Nh$ to the solution $X(t)$, if

$$E|x_h - X(T)| \leq Ch^{\gamma}, \qquad h \in (0,1),$$

where h is the step size, which divides the interval $[0,T]$ into equal length $h = \frac{T}{N}$ and $X(T)$ is the solution to the stochastic differential equation. C is a positive constant and independent of h.

3. COMPARISON BETWEEN MILSTEIN METHOD AND THE EXACT COUPLING

In this section, we present a useful comparison between two methods for solving the stochastic differential equation. Time-consuming and accurate solutions can be an effective procedure for obtaining the approximate solution for different types of methods. To give a clear overview of the methodology as a numerical implementation, we consider a two-dimensional stochastic differential equation with invertible diffusion. We apply the Milstein and the exact coupling methods on a particular SDE, so that the comparisons are made numerically. For the Milstein method, we truncate the Fourier series with specific terms, which is enough to give an accurate result. For the exact coupling method, the

diffusion is nondegenerate. For comparison purposes, we consider the following two dimension SDE:

$$\begin{aligned} dX(t) &= (\sin(Y(t)))^2 dW(t) - \frac{1}{1+X^2(t)} dV(t), \\ dY(t) &= \frac{1}{1+Y^4(t)} dW(t) + (\cos(X(t)))^2 dV(t), \\ &\text{for } 0 \le t \le 1, \text{ with } X(0) = 2 \quad \text{and} \quad Y(0) = 0. \end{aligned} \tag{11}$$

$W(t)$ and $V(t)$ are both independent standard Brownian motion. Usually, we do not know the solutions of a stochastic differential equation explicitly; therefore, we use approximate solutions to discover and compare two different methods. We calculate the error of an approximation by using the *absolute error* for the different number of steps for each method. We use the same number of simulations for both methods ($R = 10,000$). The MATLAB code for the Milstein scheme is in the following listing, which looks like the order one strong convergence in two-dimensional SDE. We compute (for example, $R = 10,000$) different Brownian paths over the interval $[0,1]$ with different step sizes. The experimental error and the elapsed time for the Milstein method are represented in the following Table 1.

Table 1. Implementation result of Milstein scheme.

	steps	step-size	absolute error	Elapsed time(hour)
1	400	0.0025	0.0691	0.124
2	800	0.0013	0.0352	0.435
3	1600	0.0006	0.0177	21.844
4	3200	0.0003	0.0090	102.797
5	6400	0.00015	0.0045	261.908

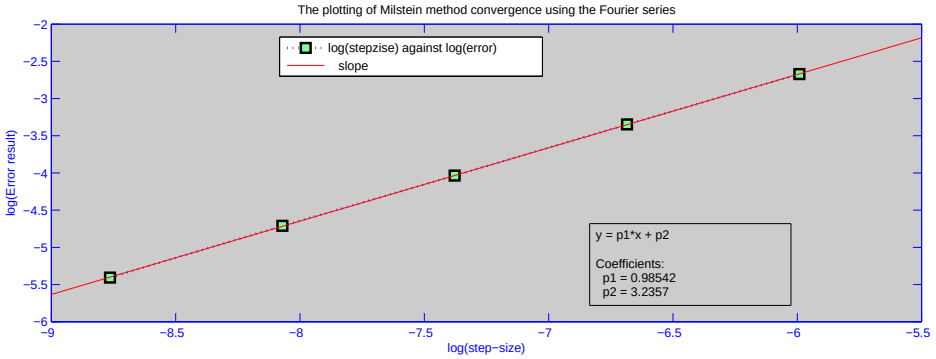

Fig. 1. The plotting of the combined method for the Milstein scheme.

The following Fig. 1 is the log plot of the absolute error with respect to the 5 different time steps. We can see that the Milstein scheme converges strongly with order one. We use 5 different step-sizes $(0.0025, 0.0013, 0.0006, 0.0003, 0.00015)$ for both methods. In the following Table 1 the experimental error and the elapsed time for the Milstein

scheme are represented. It is clear from the Table 1 and plotting result that the strong approximation error decreases as the step size decreases. The strong convergence for the exact scheme should be an order one convergence as described in Davie's paper. We run the following MATLAB code with different step sizes over a large number of paths R as follows:

```
[Error for exact coupling]
S=[400, 800, 1600, 3200, 6400];
Error=zeros(1,length(S));
for i=1:length(S)
Error(1,i)=
log(EXACTCOUPLING('YA',[1; 0],1,S(1,i)));
end
h=1./S;
fad1=log(h)
plot(log(h), Error)
```

This can be approached using a similar method to the previous Milstein method. Command (Error(1,i)=log(EXACTCOUPLING('YA',[1; 0],1,S(1,i)));) calculates the absolute value of the difference between the approximate solution x_h and the solution $X(T)$ of the SDE with different step sizes. Note also that the (bk) matrix is not invertible, but the matrix (YA) is invertible. The following Table 2 is the experimental error with respect to the 5 different time steps. The experimental error and the elapsed time for the exact coupling method are represented in the following Table 2.

Table 2. Implementation result of exact coupling method.

	steps	step-size	absolute error	Elapsed time(hour)
1	400	0.0025	0.0028	0.383
2	800	0.0013	0.0014	0.788
3	1600	0.0006	0.00069	1.545
4	3200	0.0003	0.00034	3.281
5	6400	0.00015	0.00017	6.584

The previous Fig. 2 is the log plot of the absolute error with respect to the 5 different time steps. We can see that the exact coupling method converges strongly with order one.

Comparing the results in Tables 1 and 2, we observe in both methods that the estimate of the absolute error decreases with decreasing step size. We can also observe in the previous Tables and plots that the Milstein and exact coupling methods have a strong order of convergence equal to one. We emphasize that we apply these methods over the same number of Brownian paths ($R = 10,000$) for the same step sizes. It can also be seen that the total computational time can be reduced by using the exact coupling method. We see from the Tables that there is a significant difference between the elapsed time. The Milstein code takes more than two weeks to obtain the result, but the exact coupling code takes a few hours.

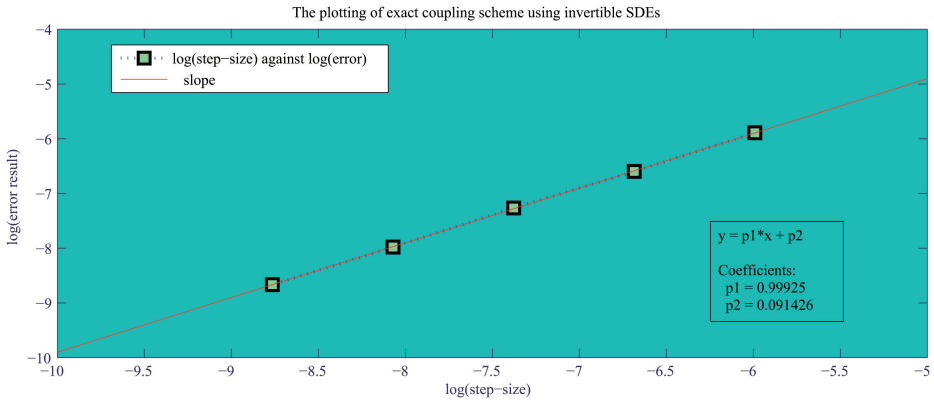

Fig. 2. The plotting of the combined method for the exact coupling.

4. CONCLUSION

Generally, we do not know the solution of the stochastic differential equation explicitly. Therefor, we use simulation to find the approximate solution and the convergence behavior. In this paper, we represented Milstein and the exact coupling methods to find the approximate solution of the stochastic differential equation. Both of these methods give an order one convergence. We implemented these schemes to stochastic differential equation for comparing the Milstein and the exact coupling methods to each other while illustrating efficiency. Additionally, we calculated error values for the Milstein and the exact coupling methods to compare the strong order and time-consuming. This article is based on MATLAB programs to show the convergence of these methods by implementations. According to our results, we can say that the exact coupling method is faster for the solution of invertible two dimensional SDE than the Milstein method. This is a very useful and efficient method for 2-dimensional SDE. However, the disadvantage of this method is that we should assume the nondegeneracy condition for the diffusion term. The advantage of the Milstein method is that there is no need for this condition, but it involves a significant computational cost. Therefore, we may conclude that the exact coupling method is more effective than the Milstein method for the invertible stochastic differential equation.

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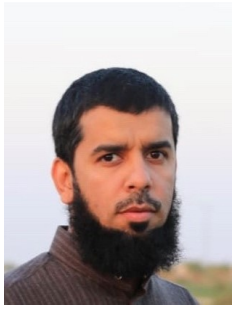

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