JOURNAL OF INFORMATION SCIENCE AND ENGINEERING 40, 253-264 (2024)
DOI: 10.6688/JISE.202403_40(2).0004

Determining the 2-Tuple Total Domination Number of a Harary Graph under Specific Degree Conditions*

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Given a graph G, a 2-tuple total dominating set is a vertex subset S such that every vertex has at least two neighbors in S. The cardinality of a minimum 2-tuple total dominating set is called the 2-tuple total domination number. In this paper, we determine the 2-tuple total domination numbers of $H_{m,n}$ for $m \in \{3,5\}$, where $H_{m,n}$ stands for a Harary graph of n vertices with degree parameter m.

Keywords: graph theory, domination, 2-tuple total domination, regular graphs, Harary graphs

1. INTRODUCTION

Given a graph G, a 2-tuple total dominating set is a vertex subset S such that every vertex has at least two neighbors in S. The cardinality of a minimum 2-tuple total dominating set is called the 2-tuple total domination number of G, denoted by $\gamma_{\times 2,t}(G)$. The notion of 2-tuple total domination was proposed by Henning and Kazemi [1].

A Harary graph is defined according to two parameters m and n, denoted by $H_{m,n}$. Assume that the n vertices, numbered from 1 to n in clockwise order, are placed evenly around a circle. If both n and m are even, then each vertex is adjacent to the $m-1$ closest ones. If n is even and m is odd, then the graph has $H_{m,n-1}$ as a subgraph, and each vertex has an additional edge connecting the diametrically opposite vertex. If both n and m are odd, each vertex is adjacent to the $m-1$ closest ones. In addition, there is an edge between vertex i and $i+\lceil n/2\rceil$, for $i<\lceil n/2\rceil$, and an edge between 1 and $\lceil n/2\rceil$.

Kazemi and Pahlavsay [2] investigated the problem of determining the 2-tuple total domination number of a Harary graph. For both n and m being even, they showed that

$$\gamma_{\times 2,t}(H_{m,n}) = \left\lceil \frac{2n}{m} \right\rceil.$$

Received October 22, 2022; accepted June 1, 2023.
Communicated by Wen-Yu Chung.
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*This research was partially supported by the National Science and Technology Council of Taiwan under contracts MOST 110-2221-E-003-003-MY3 and NSTC 112-2221-E-114-004.

For m being odd, it was shown that

$$\left\lceil \frac{2n}{m} \right\rceil \leq \gamma_{\times 2,t}(H_{m,n}) \leq \left\lceil \frac{2n}{m} \right\rceil + 1$$

when n is even, and

$$\left\lceil \frac{2n-1}{m} \right\rceil \leq \gamma_{\times 2,t}(H_{m,n}) \leq \left\lceil \frac{2n-1}{m} \right\rceil + 1$$

when n is odd. In particular, if n is even, $\gamma_{\times 2,t}(H_{m,n})$ matches the upper bound only if $1 \leq r < m/2$, where $n \equiv r \pmod{m}$; if n is odd, $\gamma_{\times 2,t}(H_{m,n})$ matches the upper bound only if $r = 0$ or $2 \leq r \leq \lceil m/2 \rceil$. Yang and Wang [3] gave an analysis for the case where both n and m are odd, with the restriction that $m \in \{3,5\}$ and $m \mid n$. In this paper, we complete the analysis for $\gamma_{\times 2,t}(H_{m,n})$ with $m \in \{3,5\}$ and arbitrary n at least $m+1$. In addition, we show that for $m = n - o(\sqrt{n})$, the 2-tuple total domination number of an m-regular graph is 3.

1.1 Related Work

Graph domination is widely investigated since it serves as an abstraction of numerous practical problems. The k-tuple total domination problem, also known as the k total domination problem, is one of the important variants. For a given graph, the problem asks for a minimum subset of vertices such that every vertex has at least k neighbors in the set. The cardinality of the requested set is the k-tuple total domination number of the graph. The concept was first proposed by Henning and Kazemi [1]. The problem is computationally hard. The complexity of computing the k-tuple total domination number on different graph classes are also conducted [4, 5]. Besides the complexity issues, combinatorial results, like the upper and lower bounds on the k-tuple total domination number of different graph classes, are also proposed, even for specific k [1–3]. Recently, related problems still draw considerable attention; *e.g.* efficient algorithms on restricted graph classes were proposed [6] and new variants were investigated [7].

2. PRELIMINARIES

For a simple graph G, a lower bound on $\gamma_{\times 2,t}(G)$ was obtained by Henning and Kazemi [1].

Theorem 1 (Henning and Kazemi [1]). *Let G be a graph with order n and maximum degree Δ. Then*

$$\gamma_{\times 2,t}(G) \geq \left\lceil \frac{2n}{\Delta} \right\rceil.$$

This is a tight lower bound. For Harary graph $H_{m,n}$ when both m and n are odd, we can slightly modify the bound in Theorem 1 as follows.

Corollary 1. *In a Harary graph $H_{m,n}$ when both m and n are odd, we have*

$$\gamma_{\times 2,t}(H_{m,n}) \geq \left\lceil \frac{2n-1}{m} \right\rceil.$$

Proof. The result can be obtained by the technique of double counting. Let S be a minimum 2-tuple total dominating set. We count the number of edges emitted from S, and that emitted from $V(H_{m,n})$ to S. Since every vertex in S is of degree m except at most one, which is of degree $m+1$, it follows that

$$|S| \cdot m + 1 \geq 2n.$$

Thus, the corollary follows. □

Remark. Note that the 2-tuple total domination number is at least 3. A graph has a 2-tuple total dominating set if and only if the minimum degree is at least 2.

Theorem 2 (Kazemi and Pahlavsay [2]). *Let $H_{m,n}$ be a Harary graph with m being odd. Let $n \equiv r \pmod{m}$. Then for even n*

$$\gamma_{\times 2,t}(H_{m,n}) = \left\lceil \frac{2n}{m} \right\rceil + 1 \implies 1 \leq r < \frac{m}{2}.$$

For odd n,

$$\gamma_{\times 2,t}(H_{m,n}) = \left\lceil \frac{2n-1}{m} \right\rceil + 1 \implies r = 0 \text{ or } 2 \leq r \leq \left\lceil \frac{m}{2} \right\rceil.$$

3. 2-TUPLE TOTAL DOMINATION IN HARARY GRAPHS

Based on Theorem 2, we complete the analyses for $\gamma_{\times 2,t}(H_{3,n})$ and $\gamma_{\times 2,t}(H_{5,n})$ by showing

- $\gamma_{\times 2,t}(H_{3,n}) = \lceil (2n-1)/3 \rceil$ for odd n (Lemma 2);
- $\gamma_{\times 2,t}(H_{3,n}) = \lceil 2n/3 \rceil$ for even n with $n \not\equiv 10 \pmod{12}$ (Lemma 2);
- $\gamma_{\times 2,t}(H_{3,n}) = \lceil 2n/3 \rceil + 1$ for $n \equiv 10 \pmod{12}$ (Lemma 3);
- $\gamma_{\times 2,t}(H_{5,n}) = \lceil (2n-1)/5 \rceil$ for odd n with $n \not\equiv 0 \pmod{5}$ (Lemma 4);
- $\gamma_{\times 2,t}(H_{5,n}) = \lceil 2n/5 \rceil + 1$ for even n with $n \equiv 1$ or $2 \pmod{5}$ (Lemma 5 and Lemma 6).

3.1 $\gamma_{\times 2,t}(H_{m,n})$ for *m*=3

First, we consider the case where $m = 3$. For a graph, a 2-coloring is called *total dominant* if the black vertices form a 2-tuple total dominating set, assuming the two colors are black and white. An *expander* for $H_{3,n}$ is one of the colored graph given in Fig. 1. We

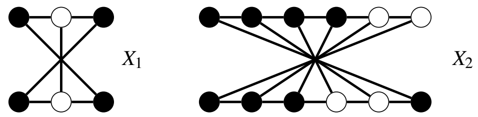


Fig. 1. Expanders for $H_{3,n}$.

denote an expander by $X\binom{t_l\ t_r}{b_l\ b_r}$, where t_l, t_r, b_l, and b_r stand for the vertices at the top-left, top-right, bottom-left, and bottom-right corners, respectively. Note that every vertex in an expander, other than the four corners, has exactly two black neighbors. A colored Harary graph $H_{3,n}$ is *expandable* through expander $X\binom{t_l\ t_r}{b_l\ b_r}$ if there exists a vertex i such that $c(i) = c(t_r)$, $c(i+1) = c(t_l)$, $c(i+\lfloor n/2\rfloor) = c(b_l)$, and $c(i+1+\lfloor n/2\rfloor) = c(b_r)$, where $c(u)$ is the color of u in the corresponding colored graph. The 4-tuple $(i, i+1, i+\lfloor n/2\rfloor, i+1+\lfloor n/2\rfloor)$ is called an *insertion quadruple*. A colored Harary graph $H_{3,n}$ is *expanded* from $H_{3,n-x}$ via expander $X\binom{t_l\ t_r}{b_l\ b_r}$ if $H_{3,n-x}$ is expandable through X and the colored $H_{3,n}$ is obtained by removing the edges $(i, i+1)$ and $(i+\lfloor n/2\rfloor, i+1+\lfloor n/2\rfloor)$, and connecting i with t_l, $i+1$ with t_r, $i+\lfloor n/2\rfloor$ with b_r, and $i+1+\lfloor n/2\rfloor$ with b_l.

Lemma 1. *Let $H_{3,n}$ be a colored Harary graph that is expanded from a colored $H_{3,n-x}$ via expander X, where $|X| = x$. If there are $\lceil(2(n-x)-1)/3\rceil$ black vertices in $H_{3,n-x}$, then there are exactly $\lceil(2n-1)/3\rceil$ black vertices in $H_{3,n}$. Moreover, the colored $H_{3,n}$ is expandable through X.*

Proof. Let b be the number of black vertices in X. The pair (x,b) is either $(6,4)$ or $(12,8)$. In either case, it can be verified that

$$b = \left\lceil \frac{2n-1}{3} \right\rceil - \left\lceil \frac{2(n-x)-1}{3} \right\rceil .$$

To show that the colored $H_{3,n}$ is expandable through X, we claim that there is an insertion point. For $X = X_1$, the four corners are all black, which implies that vertices in the insertion quadruple $(i, i+1, i+\lceil n/2\rceil, i+1+\lceil n/2\rceil)$ are all colored black. Then $(i, t_l, i+\lceil n/2\rceil, b_r)$ can serve as the insertion quadruple for the colored $H_{3,n}$. For $X = X_2$, with a proper renumbering, the middle four vertices, *i.e.* the black-black pair on the top row and the black-white pair on the bottom row, can serve as the insertion quadruple. □

By Lemma 1, the process of "expanding an existing coloring" using an expander can be applied iteratively. It can be verified that in the expanded Harary graph, every vertex in the expander has exactly two black neighbors, and the remaining vertices have the same black neighbors. Thus, to show that $\gamma_{\times 2,t}(H_{3,n})$ matches the lower bound of $\lceil(2n-1)/3\rceil$ for odd n, it suffices to give total dominant colorings for some basis instances.

Lemma 2. *Let n be a positive integer with $n \geq 4$. Then*

$$\gamma_{\times 2,t}(H_{3,n}) = \left\lceil \frac{2n-1}{3} \right\rceil$$

if n is odd, and

$$\gamma_{\times 2,t}(H_{3,n}) = \left\lceil \frac{2n}{3} \right\rceil$$

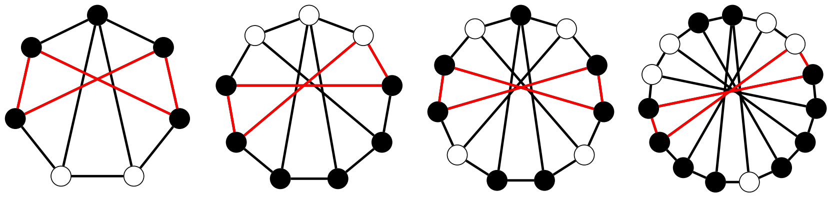


Fig. 2. Total dominant colorings for $H_{3,n}$, for $n \in \{7,9,11,15\}$. Red lines mark the place to insert the expander.

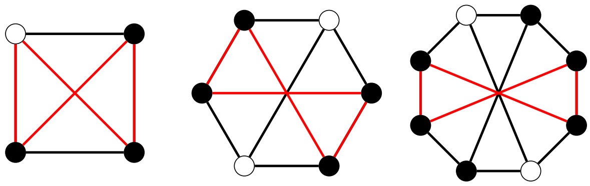


Fig. 3. Total dominant colorings for $H_{3,n}$, for $n \in \{4,6,8\}$. Red lines mark the place to insert the expander.

if n is even and $n \not\equiv 10 \pmod{12}$.

Proof. Every odd number n satisfies one of the following: (i) $n \equiv 1 \pmod 6$; (ii) $n \equiv 5 \pmod 6$; (iii) $n \equiv 3 \pmod{12}$; (iv) $n \equiv 9 \pmod{12}$. The basis instances have n equal to 7, 11, 9, and 15, respectively. The total dominant coloring for each basis instance is given in Fig. 2. Clearly, both $H_{3,7}$ and $H_{3,11}$ are expandable through X_1, and both $H_{3,9}$ and $H_{3,15}$ are expandable through X_2. The remaining instance is $H_{3,5}$, which can be easily verified that $\gamma_{\times 2,t}(H_{3,5}) = \lceil \frac{2 \cdot 5 - 1}{3} \rceil = 3$.

Similarly, for even n with $n \not\equiv 10 \pmod{12}$, we have (i) $n \equiv 4 \pmod{12}$; (ii) $n \equiv 0 \pmod 6$; (iii) $n \equiv 2 \pmod 6$. Basis instances are shown in Fig. 3. Each of them are expandable through either X_1 or X_2. □

Lemma 3. *For $n \geq 10$ and $n \equiv 10 \pmod{12}$, $\gamma_{\times 2,t}(H_{3,n}) = \lceil 2n/3 \rceil + 1$.*

Proof. Let $n = 12k + 10$ for $k \geq 0$. We prove the lemma by induction on k. It can be easily verified that $\gamma_{\times 2,t}(H_{3,10}) = 8$. For any $k \geq 1$, suppose to the contrary that $\gamma_{\times 2,t}(H_{3,n}) = \lceil 2n/3 \rceil = 8k + 7$, and we associate the $H_{3,n}$ with a total dominant coloring with $8k+7$ black vertices. Let D be the set of black vertices. Since $H_{3,n}$ is 3-regular and every vertex has at least two neighbors in D, it can be derived that each vertex in $H_{3,n}$ has exactly two neighbors in D except one, say x, such that $|N(x) \cap D| = 3$. In addition, since the sum of degrees in the subgraph induced on D is even, we have $x \notin D$. Without loss of generality we assume that $x = n/2 + 1$. Consider the clockwise neighbors $N_1 = \{x+1, x+2, \ldots, x+6\}$ of x and their diametrically opposite neighbors $N_2 = \{x+1-n/2, x+2-n/2, \ldots, x+6-n/2\}$. There is exactly one feasible way to color these vertices; namely N_1 and N_2 correspond to the top row and bottom row of expander X_2 (Fig. 1), respectively. In addition, each of $x+7$, $x+7-n/2$, and $x-n/2$ is colored black, which shows that this colored $H_{3,n}$ is expanded from a colored $H_{3,n-12}$, containing exactly $8(k-1)+7$

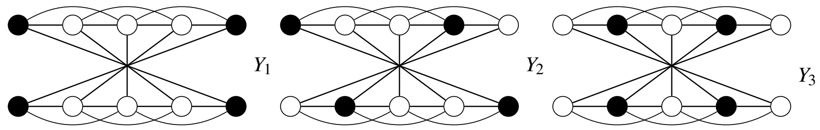


Fig. 4. Expanders for $H_{5,n}$.

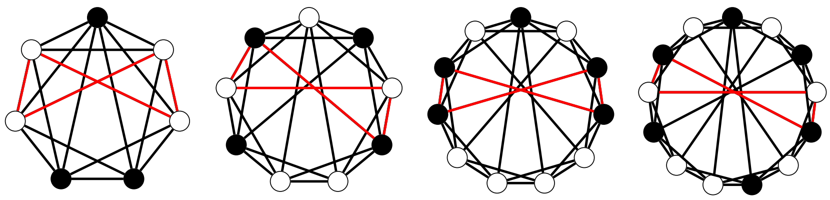


Fig. 5. Total dominant colorings for $H_{5,n}$, for $n \in \{7,9,11,13\}$. Red lines mark the place to insert the expander.

black vertices. By the induction hypothesis we have a contradiction, and the lemma is proved. □

3.2 $\gamma_{\times 2,t}(H_{m,n})$ for $m = 3$ and odd n

A similar argument to the case where $m = 3$ can be applied to show $\gamma_{\times 2,t}(H_{5,n}) = \lceil (2n-1)/5 \rceil$ for $n \not\equiv 0 \pmod 5$ when n is odd. Note that when applying the expanding process, in an expander not only the single vertex at each corner has to be consider, but also the one next to each corner. In the graph to expand, one has to consider an octuple instead of a quadruple; namely to consider whether the colors of the eight vertices closest to the corners match the colors of the corresponding vertices in the expander.

There are three expanders, as shown in Fig. 4, while the total dominant colorings of the basis instances are given in Fig. 5.

Lemma 4. *Let n be an odd number with $n \geq 7$ and $n \not\equiv 0 \pmod 5$. Then*

$$\gamma_{\times 2,t}(H_{5,n}) = \left\lceil \frac{2n-1}{5} \right\rceil.$$

Proof. Every odd number n with $n \geq 7$ and $5 \nmid n$ satisfies one of the following: (i) $n \equiv 1 \pmod{10}$; (ii) $n \equiv 3 \pmod{10}$; (iii) $n \equiv 7 \pmod{10}$; (iv) $n \equiv 9 \pmod{10}$. The basis instances are $H_{5,11}$, $H_{5,13}$, $H_{5,7}$, and $H_{5,9}$, respectively. The dominant colorings are shown in Fig. 5. It can be verified that the colored $H_{5,7}$ is expandable through expander Y_1, the colored $H_{5,9}$ and $H_{5,13}$ are expandable through expander Y_2, and the colored $H_{5,11}$ is expandable through expander Y_3. □

Remark. For $5 \mid n$, $\gamma_{\times 2,t}(H_{5,n}) = \lceil (2n-1)/5 \rceil + 1$, as shown by Yang and Wang [3].

3.3 $\gamma_{\times 2,t}(H_{m,n})$ $m = 5$ and even n

In this section, we show that $\gamma_{\times 2,t}(H_{5,n}) = \lceil 2n/5 \rceil + 1$ for even n with $n \equiv 1$ or $2 \pmod 5$. By Theorem 1 we have the following corollary.

Corollary 2. *Let D be a 2-tuple total dominating set of $H_{5,n}$ such that $|D| = \lceil 2n/5 \rceil$. Given $n \equiv r \pmod 5$ for $r \in \{1,2\}$, we have*

$$n_{\neq 2} := |\{u \in V : |N(u) \cap D| \neq 2\}| \leq 3. \tag{1}$$

Moreover, the following assertions hold:

(a) *if $r = 2$, then $n_{\neq 2} = 1$, and there is a unique vertex $x \in V \setminus D$ such that $|N(x) \cap D| = 3$;*

(b) *if $r = 1$, then one of the following holds:*

(b.1) *$n_{\neq 2} = 1$, and there is a unique vertex $x \in V \setminus D$ such that $|N(x) \cap D| = 5$;*

(b.2) *$n_{\neq 2} = 2$, and there are two vertices x and y such that $|N(x) \cap D| = 3$, $|N(y) \cap D| = 4$, and $x \in V \setminus D$.*

(b.3) *$n_{\neq 2} = 3$, and there are three vertices x, y, and z such that $|N(x) \cap D| = |N(y) \cap D| = |N(z) \cap D| = 3$. In particular, either $\{x,y,z\} \subseteq V \setminus D$, or $x \in V \setminus D$ and $\{y,z\} \subseteq D$.*

To ease the presentation, in the remainder of the section we write the number of vertices as $2n$.

Lemma 5. *For $2n > 5$ and $2n \equiv 2 \pmod 5$, $\gamma_{\times 2,t}(H_{5,2n}) = \lceil 4n/5 \rceil + 1$.*

Proof. By Theorem 2, it suffices to show that $\gamma_{\times 2,t}(H_{5,2n}) \neq \lceil 4n/5 \rceil$. Suppose to the contrary that $\gamma_{\times 2,t}(H_{5,2n})$ matches the lower bound $\lceil 4n/5 \rceil$. By Corollary 2 (a), for any minimum 2-tuple total dominating set D, there is a unique vertex x with $|N(x) \cap D| = 3$ and $x \in V \setminus D$. Without loss of generality, let $x = 1$. By symmetry we develop the cases for $N(x) \cap D$: $\{n+1, 2n\} \subseteq N(x) \cap D$ (Fig. 6 (a)); $N(x) \cap D = \{3, n+1, 2n-1\}$ (Fig. 6 (b)); $N(x) \cap D = \{2, 2n-1, 2n\}$ (Fig. 6 (c)); $N(x) \cap D = \{2, 3, 2n-1\}$ (Fig. 6 (d)). In either case, it can be derived that there is a vertex other than x having three neighbors in D, which leads to a contradiction. Details are given in Figs. 6 (a)-(d). □

Lemma 6. *For $2n > 5$ and $2n \equiv 1 \pmod 5$, $\gamma_{\times 2,t}(H_{5,2n}) = \lceil 4n/5 \rceil + 1$.*

Proof. By Theorem 2, it suffices to show that $\gamma_{\times 2,t}(H_{5,2n}) \neq \lceil 4n/5 \rceil$. Suppose to the contrary that $\gamma_{\times 2,t}(H_{5,2n}) = \lceil 4n/5 \rceil$. By Corollary 2(b), for a minimum 2-tuple total dominating set D we develop the following cases:

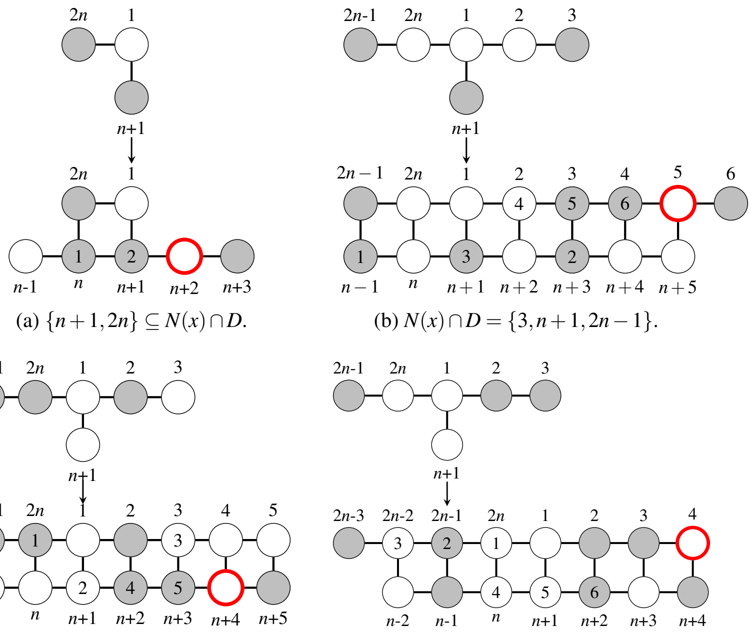
i $\exists x \in V \setminus D$ such that $|N(x) \cap D| = 5$;

ii $\exists x \in V \setminus D$ and $y \in D$ such that $|N(x) \cap D| = 3$ and $|N(y) \cap D| = 4$;

iii $\exists \{x,y\} \subseteq V \setminus D$ such that $|N(x) \cap D| = 3$ and $|N(y) \cap D| = 4$;

iv $\exists x \in V \setminus D$ and $\{y,z\} \subseteq D$ such that $|N(x) \cap D| = |N(y) \cap D| = |N(y) \cap D| = 3$;

v $\exists \{x,y,z\} \subseteq V \setminus D$ such that $|N(x) \cap D| = |N(y) \cap D| = |N(y) \cap D| = 3$.



(a) $\{n+1, 2n\} \subseteq N(x) \cap D$. (b) $N(x) \cap D = \{3, n+1, 2n-1\}$.

(c) $N(x) \cap D = \{2, 2n-1, 2n\}$. (d) $N(x) \cap D = \{2, 3, 2n-1\}$.

Fig. 6. Possible 2-tuple total dominating sets of size $\lceil 4n/5 \rceil$ for $H_{5,2n}$, where $2n \equiv 2 \pmod 5$. Numbers inside the vertices indicate the order with which the colors for their neighborhoods are determined. Red circles indicate the vertices violating the corresponding condition given in Corollary 2.

For each case, a contradiction can be derived in a straightforward manner once $\{x, y, z\}$ is specified. We show how the triple is chosen and provide illustrations for two of them (Figs. 7 and 8) in the appendix.

For (i) we assume that $x = 1$. Every neighbor of x is in D, and it can be derived that $N(n+1) \cap D = \emptyset$.

For (ii), we assume that $y = 1$. By Corollary 2(b) one of 2 and $2n$ is not in D since otherwise there is a vertex in D having three neighbors in D. Thus, it suffices to consider $N(y) \cap D = \{2, 3, n+1, 2n-1\}$, which results in $|N(2n) \cap D| = |N(n+3) \cap D| = 3$.

For (iii), we assume that $y = 1$ and consider $N(y) \cap D$ to be $\{2, 3, n+1, 2n\}$; $\{2, 3, n+1, 2n-1\}$; $\{2, 3, 2n-1, 2n\}$.

For (iv), we assume that $x = 1$ and consider $N(x) \cap D$ to be $\{n+1, 2n-1, 2n\}$; $\{2, n+1, 2n\}$; $\{2, n+1, 2n-1\}$; $\{3, n+1, 2n-1\}$ (Fig. 7); $\{2, 2n-1, 2n\}$; $\{3, 2n-1, 2n\}$.

For (v), we assume that $x = 1$ and consider $N(x) \cap D$ to be $\{2, 3, n+1\}$; $\{2, n+1, 2n\}$; $\{3, n+1, 2n\}$; $\{3, n+1, 2n-1\}$; $\{2, 2n-1, 2n\}$; $\{3, 2n-1, 2n\}$ (Fig. 8). Note that in this case an additional assumption $|N(n+1) \cap D| = 2$ can be made. The reason is that there are exactly three vertices which have more than two neighbors in D. By the pigeonhole principle we may choose x as the vertex satisfying the assumption.

In either case, it can be verified that the corresponding condition in Corollary 2(b) is not satisfied, and thus no 2-tuple total dominating set of the requested size exists. $\square$

3.4 For Large *m*

We determine the 2-tuple total domination number for Harary graph when m is small. Here we show that when m is large enough, the exact value of the 2-tuple total domination number can also be determined.

Theorem 3. *Let G be an m-regular graph of order n. If $n-m=o(\sqrt{n})$, then $\gamma_{\times 2,t}(G)=3$.*

Proof. We prove the theorem by a simple probabilistic argument. Let S be a 3-vertex subset chosen uniformly at random, *i.e.* with probability $p=1/\binom{n}{3}$, and let X be the event that S is not a 2-tuple total dominating set. We claim that $\Pr(X)<1$ when $n-m=o(n)$.

Let X_v be the event in which $|N(v)\cap S|<2$. Then

$$\Pr(X_v)=p\left(\binom{n-m}{3}+\binom{m}{1}\binom{n-m}{2}\right).$$

Let $k=n-m$. By the union bound, we have

$$\begin{aligned}\Pr(X)=\Pr\left(\bigcup_{v\in V(G)} X_v\right)&\le\sum_{v\in V(G)}\Pr(X_v)\\&=np\left(\binom{k}{3}+\binom{n-k}{1}\binom{k}{2}\right)=\frac{-2k^3+3nk^2+(2-3n)k}{(n-1)(n-2)}.\end{aligned}$$

It follows that $\Pr(X)<1$ when $k=o(\sqrt{n})$, which implies $\gamma_{\times 2,t}(G)\le 3$. Along with the lower bound of 3 on the size of a 2-tuple total dominating set, the theorem is proved. □

Remark. We note here that Theorem 3 can be applied to Harary graphs when n is even.

4. CONCLUDING REMARKS

In this paper, we complete the analysis on $\gamma_{\times 2,t}(H_{m,n})$ for $m\in\{3,5\}$. When $H_{m,n}$ is regular, we also give the exact value of $\gamma_{\times 2,t}(H_{m,n})$ when m is large enough. The same idea can be applied to derive $\gamma_{\times 2,t}(H_{m,n})$ for some small m. However, such a process would be tedious even for $m=7$, and is difficult to be extended when m becomes large. To tackle the analysis on $\gamma_{\times 2,t}(H_{m,n})$ for arbitrary m, properties to make the analysis succinct are necessary.

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A. DETAILED ANALYSES FOR THE PROOF OF LEMMA 6

We provide detailed analyses for the cases given in the proof of Lemma 6. In the figures, numbers inside the vertices indicate the order with which (some) colors for their neighborhoods are determined. Red circles indicate the vertices violating the corresponding condition given in Corollary 2.

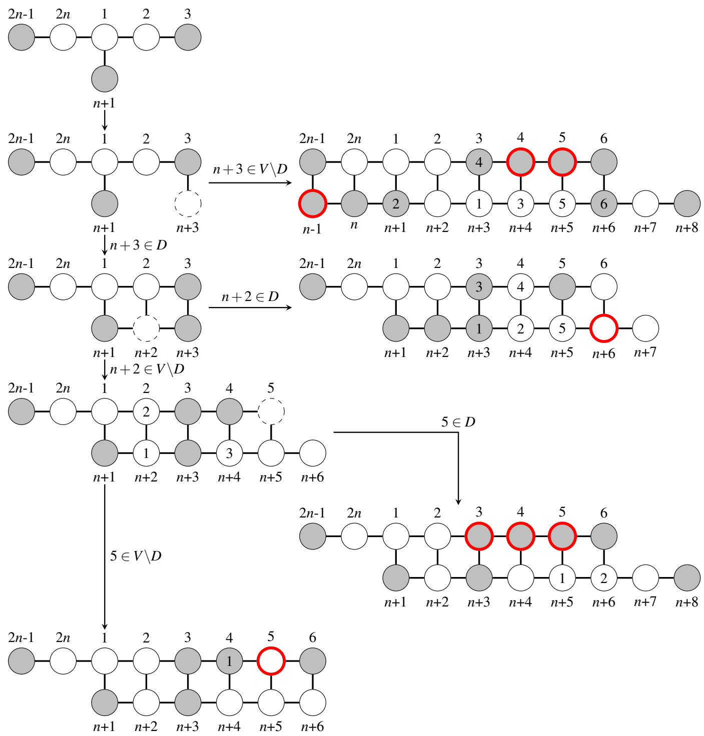

Fig. 7. An analysis for $|N(x) \cap D| = |N(y) \cap D| = |N(z) \cap D| = 3$ with $x = 1 \in V \setminus D$ and $\{y,z\} \subseteq D$. Assume that $N(x) = \{3, n+1, 2n-1\}$.

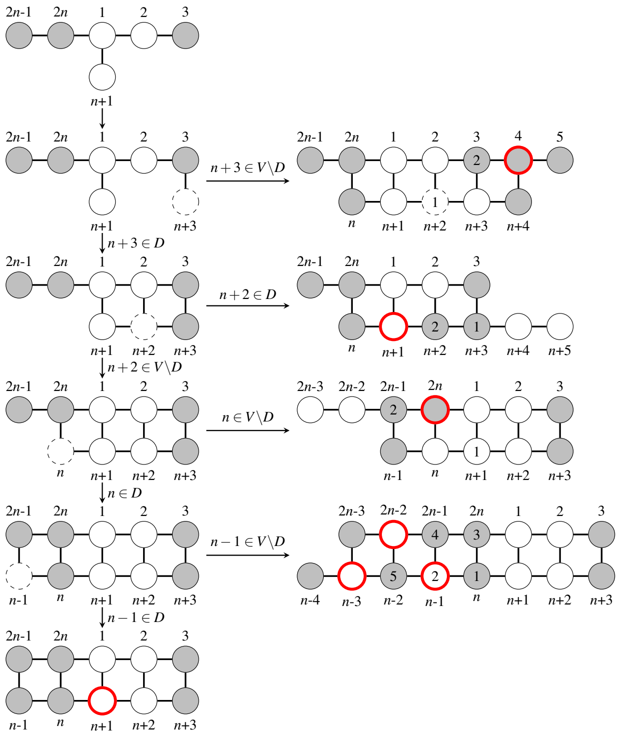

Fig. 8. An analysis for $|N(x) \cap D| = |N(y) \cap D| = |N(z) \cap D| = 3$ with $\{x,y,z\} \subseteq V \setminus D$. Assume that $x = 1$ and $|N(n+1) \cap D| = 2$, with $N(x) = \{3, 2n-1, 2n\}$.

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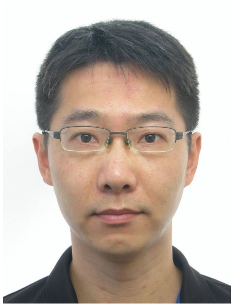

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