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Short Paper

Linear Tail-Biting Trellis and Its Sectionalization

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In this paper, we discuss the sectionalization in linear tail-biting trellis. In [3], Lafourcade and Vardy investigated the basic property of sectionalization in conventional trellis. Their proofs of basic properties are based on conventional trellis structure which fails to be generalized to linear tail-biting trellis. We derive our proofs from the construction of Koetter-Vardy (KV) trellis in [1]. All properties of sectionalization in conventional trellis are preserved in linear tail-biting trellis. Especially, we prove the necessary condition of an optimal sectionalization in linear tail-biting trellis.

Keywords: information security, coding, cryptography, trellis, sectionalization

1. INTRODUCTION

Trellis representations of linear block codes have received much attention in recent years [3, 7, 9]. Such representations not only illuminate code structure, but also often lead to efficient trellis-based decoding algorithms. For any linear block code, there exists a unique, up to isomorphism, minimal conventional trellis. Furthermore, we can efficiently construct the minimal conventional trellis for any linear code from its generator matrix or parity-check matrix by several methods, such as Bahl-Cocke-Jelinek-Raviv (BCJR), Massey, Forney, and Kschischang-Sorokine constructions [4, 5, 8]. Hence, the theory on conventional trellis is well developed. However, much less is known about tail-biting trellises. We have to constrain the investigation on linear tail-biting trellis. After Mc-Eliece [6] introduced the class of simple linear trellises in 1996, R. Koetter and A. Vardy developed the general theory of linear tail-biting trellis in [1, 2].

To minimize the complexity of a conventional trellis, Lafourcade and Vardy [3] consider an operation on the axis, called sectionalization, which can drastically change

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the structure of trellises. A lot of examples show that proper sectionalization can make the decoding much less complicated. The question should be raised naturally: is the sectionalization in linear tail-biting trellis as good as it in conventional trellis? Our goal in this work is to lay out some of the foundations of the theory.

The remainder of this paper is organized as follows. We first prove that two operations, amalgamation and product, are commutative. They first appear in [3] and [4]. Thanks to the KV construction, we show that some properties proved in [3] for conventional trellis are preserved for linear tail-biting trellis. Section 2 introduces some preliminaries and backgrounds. Section 3 presents our main results of sectionalization for linear tail-biting trellis. Finally, Section 4 concludes this paper.

2. PRELIMINARY

We first introduce some basic notions on linear tail-biting trellises for block codes, which we borrow from [1] and [2]. For more details, the readers can refer to the two references.

An edge-labeled directed graph is a triple (V, E, A), consisting of a set V of vertices, a finite set A called the alphabet, and a set E of ordered triples (v, a, v'), with $v, v' \in V$ and $a \in A$, called edges. We say that an edge (v, a, v') begins at v, and ends at v', and has label a. We only give the definition of a tail-biting trellis, and omit the definition of a conventional trellis, which is similar to the definition of a tail-biting trellis.

Definition 1: A tail-biting trellis $T = (V, E, A)$ of depth n is an edge-labeled directed graph with the following property: the vertex set V can be partitioned as

$$V = V_0 \cup V_1 \cup \ldots \cup V_n \qquad (1)$$

such that every edge in T begins at a vertex of V_i and ends at a vertex of V_{i+1}, for some $i = 0, 1, \ldots, n-1$, and $V_0 = V_n$. The set E of edges is partitioned in a natural way as $E = E_1 \cup E_2 \cup \ldots \cup E_n$, where E_i is the set of all edges ending at a vertex of V_i. The sets $V_0, V_1, \ldots, V_n$ are called the vertex class of T. The ordered index set $I = \{0, 1, \ldots, n\}$ induced by the partition in Eq. (1) is called the time axis for T.

A cycle of length n in a tail-biting trellis T is a closed path in T through n distinct vertices. Clearly, any cycle in T contains exactly one vertex in each vertex class. A tail-biting trellis T is reduced if any vertex and edge belong to at least one cycle. The set of edge labels along a cycle in T is an n-tuple $(a_0, a_1, \ldots, a_{n-1})$ over the label alphabet A. Postulating that all cycles in T start at a vertex of V_0, every cycle defines a vector $(a_0, a_1, \ldots, a_{n-1}) \in A^n$, which is called an edge-label sequence in T. Let $C(T)$ denote the set of all edge-label sequences in T. Then $C(T)$ is called the edge-label code of T. T is a tail-biting trellis for the block code C over A if $C(T) = C$. If every vertex in each vertex class V_i, $0 \leq i \leq n-1$, is labeled by a sequence of length ι_i over A, where $\iota_i \geq \lceil \log_{|A|}^{|V_i|} \rceil$, then this kind of trellis is called a labeled trellis. Let $\iota = \iota_0 + \iota_1 + \iota_{n-1}$. Then every cycle Γ in a labeled tail-biting trellis defines an ordered sequence of length $n + \iota$ over A, consisting of the labels of edges and vertices in Γ. We refer to such a sequence as a label sequence in T. Let $S(T)$ denote the set of all such label sequences. $S(T)$ is called the label code of T. We always assume that the alphabet set A is a finite field F_q. A labeled trellis

$T = (V, E, F_q)$ is linear over F_q if T is reduced and $S(T)$ is a linear code over F_q. An unlabeled trellis T is said to be linear if there exists a vertex labeling of T such that the resulting labeled trellis is linear.

It is proved in [2] that a linear trellis can be factored as $T = T_{x_1} \times T_{x_2} \ldots \times T_{x_k}$, where $T_{x_1}, T_{x_2}, \ldots, T_{x_k}$ are elementary trellises. According to [1], there is a characteristic matrix χ for C which can be obtained by certain algorithms. χ often contains n characteristic generator $(x, [x])$.

For $i, j \in I = \{0, 1, \ldots, n-1\}$, define closed cyclic interval $(i, j]$ as follows:

$$(i,j] = \begin{cases} \{i, i+1, \ldots, j\} & \text{if} \quad i \le j \\ \{i, i+1, \ldots, n-1, 0, \ldots, j\} & \text{if} \quad i > j \end{cases}. \tag{2}$$

We also define the semi-open cyclic interval $(i, j]$ as $[i, j]\backslash\{i\}$. Let C be any linear code with length n over the finite field F_q, *i.e.*, $C \subseteq F_q^n$. For a nonzero element $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, a cyclic interval $(i, j]$ is called a span of c if $(i, j]$ contains all nonzero elements of c, and denoted by $[c] = (i, j]$. If $i \le j$, we call $(i, j]$ a linear span. Otherwise $(i, j]$ is called a circular span.

The out-degree vector $e_i^{out} = (e_{i0}^{out}, \ldots e_{i,n-1}^{out})$ and in-degree vector $e_i^{in} = (e_{i1}^{in}, \ldots e_{in}^{in})$ are defined for elementary trellis in KV construction where e_{il}^{in} and e_{il}^{out} are the logarithm of the indegree and outdegree of node in V_l of elementary trellis T_{x_i} in the base of q. If $x_i \in C$ and $[x_i] = (a, b]$, then $e_{ia}^{out} = 1$, $e_{ij}^{out} = 0$ for $j \ne a$ and $e_{ib}^{in} = 1$, $e_{ij}^{in} = 0$ for $j \ne b$.[1]

By the property of KV construction [1] of linear tail-biting trellis, $T = T_{x_1} \times T_{x_2} \ldots \times T_{x_k}$, for each time i, its outdegree can be obtained by multiplying each elementary trellis involved in KV construction of T. That is, if (v', v'') is a vertex in the trellis product $T' \times T''$ then

$$deg_{in}(v', v'') = deg_{in}(v') \cdot deg_{in}(v'')$$
$$deg_{out}(v', v'') = deg_{out}(v') \cdot deg_{out}(v'')$$

Generalize it to k trellis $T = T_{x_1} \times T_{x_2} \ldots \times T_{x_k}$. Node $(v_1, \ldots, v_k) \in T$, we have

$$deg_{in}(v_1, \ldots, v_k) = \prod_{i=1}^{k} deg_{in}(v_i) \tag{3}$$

$$deg_{out}(v_1, \ldots, v_k) = \prod_{i=1}^{k} deg_{out}(v_i) \tag{4}$$

We now define the operations of composition and amalgamation of trellises. Given a trellis $T = (V, E, A)$ of depth ν and a trellis $T' = (V', E', A')$ of depth ν', such that $V_\nu = V'_0$, we can simply "glue" them together to form a trellis of depth $\nu + \nu'$. We use $T^\circ = T \circ T'$ to denote composition. T° is given by

$$V_i^\circ = \begin{cases} V_i, & i = 0, 1, \ldots \nu, \\ V'_{i-\nu}, & i = \nu + 1, \ldots, \nu + \nu'. \end{cases} \tag{5}$$

Their edge set is given by $E^\circ = E \cup E'$, edge-labels at time i in T° is given by $A_i^\circ = A_i$,

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[1] V_0 has no indegree and V_n has no outdegree, our definition makes sense in this situation.

$i = 0, 1, \ldots, \nu$ and $A_i^{\circ} = A'_{i-\nu}$, $i = \nu + 1, \ldots, \nu + \nu'$.

Instead of composition, we can amalgamate two trellises into one.

Definition 2: *A* trellis $T^* = (V^*, E^*, A^*)$ of depth one is said to be the amalgamation of T and T', if $V_0^* = V_0$, the edge-label alphabet given by $A^* = A_1 \times A_2 \ldots \times A_\nu \times A'_1 \times A'_2 \ldots \times A'_{\nu'}$. E^* is the set of paths from V_0 to $V'_{\nu'}$ in $T \circ T'$. There is an edge (v_1, v_2, α) in E^* if and only if a path labeled α from v_1 to v_2 in $T \circ T'$. Clearly, composition operation is associative.

In general, the amalgamation of trellis T is obtained by first composing T into n trellis with length one and then amalgamating them.

Definition 3: T is a trellis with length n. $T = T_1 \circ T_2 \circ \ldots \circ T_n$ where T_i is a trellis with length one. The amalgamation of T denoted by $\tilde{T}$ is given by

$$\tilde{T} = T_1 * T_2 * \cdots * T_n. \tag{6}$$

Next, we introduce the objective function that satisfies some certain properties.

Definition 4: $\mathcal{T}$ is the set of trellises. Function F: $\mathcal{T} \rightarrow N$ is said to be decomsition-linear, if for all trellis $T \in \mathcal{T}$, any decomposition $T = T_1 \circ T_2$, we have $F(T_1 \circ T_2) = F(T_1) + F(T_2)$.

For linear tail-biting trellis, the Viterbi decoding complexity of T is approximately given by:

$$D(T) = h(2|E| - |V| + |V_0|). \tag{7}$$

h is the number of state that intermediate node need to memorize, since we require that Viterbi algorithm runs for each root node which produces h candidate code. In the other word, h is the number of root node of the entire tail-biting trellis instead of the root node of a section of it. In this sense, h is constant. For example, $T_{h,h'} = T_{h+1} \circ T_{h+2} \circ \ldots \circ T_{h'}$ is a section of tail-biting trellis $T = T_1 \circ T_2 \circ \ldots \circ T_n$, where T_i is a trellis with length one. Then the Viterbi decoding complexity of $T_{h,h'}$ is

$$D(T) = h(2|E| - \sum_{i=h}^{h'-1} |V_i|) = |V_0|(2|E| - \sum_{i=h}^{h'-1} |V_i|). \tag{8}$$

It follows that $D(T)$ is decomposition-linear. It was shown in [3] that the function $M(T)$ which counts the total number of operations has the following property that:

$$M(T_1 \circ T_2) = M(T_1) + M(T_2),$$
$$M(T_1 * T_2) \geq M(T_1) + M(T_2).$$

Let $F(T) = D(T) + M(T)$. The optimal sectionalization with respect to F is to minimize the value of F under all possible composition and amalgamation. Without proof, we give the following lemma which appears as well in [3].

Lemma 1: Suppose that a section $T^* = T' * T''$ such that $|E| + |E''| \leq |E|$. Then T^* can not be a section in an optimal sectionalization with respect to objective function $F(T) = D(T) + M(T)$.

3. MAIN RESULT

In this section, we will generalize sectionalization to linear tail-biting trellis which is a general case of conventional trellis. However, the proof of sectionalization in conventional trellis is not valid for linear tail-biting trellis since p_i, f_i is not well-defined. Our proof derives from the construction of Koetter-Vardy (KV) [1] trellis.

Lemma 2: T_1 and T_2 are two trellis, let T denote their product, $T = T_1 \times T_2$. The amalgamation of T generates a new trellis which is called $\widetilde{T}$. The amalgamation of T_1 and T_2 is denoted by $\widetilde{T}$ and $\widetilde{T}_2$ respectively. Then

$$\widetilde{T} = \widetilde{T}_1 \times \widetilde{T}_2 \tag{9}$$

Proof: First, we prove that each side in Eq. (9) has the same vertex set. Without loss of generality, T_1 and T_2 have length n. Root node set and toor node set of T_1 denoted by V_0^1, V_n^1. Clearly, $\widetilde{T}_1$ shares the same root node and toor node with T_1. V_0^2 and V_n^2 are defined for T_2 and $\widetilde{T}_2$. Then the root node set and toor node set for $\widetilde{T}$ are $V_0^1 \times V_0^2$ and $V_n^1 \times V_n^2$ which are the same of $\widetilde{T}_1 \times \widetilde{T}_2$.

Next, we will show that each pair of node (root node and toor node) in both side has the edge with same label. Now, assume that there is an edge label α from root node (v_0^1, v_0^2) to toor node (v_n^1, v_n^2) in $\widetilde{T}$. By definition of amalgamation operation, we derive a path $P = (((v_0^1, v_0^2), (v_1^1, v_1^2)), \ldots, (v_n^1, v_n^2))$ in T. Let α_1 be the path $(v_0^1, v_1^1, , \ldots, v_n^1)$ labeled in T_1 and α_2 be $(v_0^2, v_1^2, , \ldots, v_n^2)$ in T_2. Thus, the edge from v_0^1 to v_n^1 labeled α_1 appears in $\widetilde{T}_1$. Similarly, the edge from v_0^2 to v_n^2 labeled α_2 appears in $\widetilde{T}_2$. By definition of product operation, $\alpha = \alpha_1 + \alpha_2$ appears in $\widetilde{T}_1 \times \widetilde{T}_2$ as an edge from (v_0^1, v_0^2) to (v_n^1, v_n^2). The proof process can be reversed. Therefore, we complete our proof. ❑

Lemma 2 shows that amalgamation and product are commutative between two trellis. Next corollary will generalize it to n trellises.

Corollary 1: $T_1, \ldots, T_n$ are n trellises. Their amalgamation is denoted by $\widetilde{T}_1, \ldots \widetilde{T}_n$. Then,

$$(\widetilde{T_1 \times T_2 \ldots \times T_n}) = \tilde{T}_1 \times \tilde{T}_2 \ldots \times \tilde{T}_n. \tag{10}$$

Proof: By Lemma 2, we have

$$\begin{aligned}(\widetilde{T_1 \times T_2 \ldots \times T_n}) &= \widetilde{T}_1 \times (\widetilde{T_2 \ldots \times T_n}) \\ &= \ldots \\ &= \widetilde{T}_1 \times \widetilde{T}_2 \ldots \times \widetilde{T}_n.\end{aligned}$$ ❑

Next proposition gives the relation of in-degree vector and out-degree vector in elementary trellis before and after amalgamation.

Proposition 1: If $T_{h,h'} = T_{h+1} \circ T_{h+2} \circ \ldots \circ T_{h'}$ is a section of elementary trellis $T = T_1 \circ T_2 \circ \ldots \circ T_n$, the amalgamationy $\widetilde{T}_{h,h'}$ of $T_{h,h'}$ show that its indegree and outdegree are $q^{\sum_{j=h+1}^{h'} e_j^{in}}$

and $q^{\sum_{j=h}^{h'-1} e_j^{out}}$ respectively where e^{in} and e^{out} are in-degree and out-degree vector of T.

Proof: Since the elementary trellis can be divided into two kinds according to its linear or circular span. Therefore amalgamation of $T_{h,h'}$ generates six types which exhibits in Fig. 1. Left side of the figure is $T_{h,h'}$ and right side is $\widetilde{T}_{h,h'}$. It is trivially true. ❑

Corollary 1 tells us that product and amalgamating operation are commutative. The proof of [3] in characterizing necessary condition for optimal sectionalization is not valid for linear tail-biting trellis. To approach next proposition, we will resort to KV construction of linear tail-biting trellis, together with our Corollary 1 and Proposition 1.

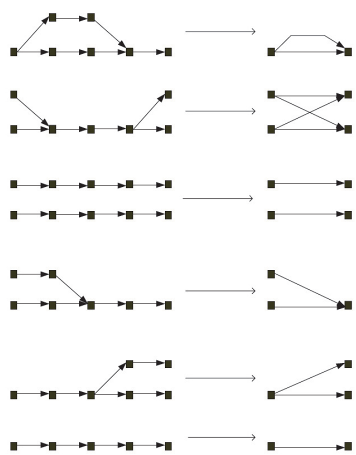
Fig. 1. Six types of amalgamation in section of elementary trellis.

Proposition 2: If $T_{h,h'} = T_{h+1} * T_{h+2} \ldots * T_{h'}$ is a section in any sectionalization of linear tail-biting trellis T, then the number of edges in $T_{h,h'}$ is lower bounded by:

$$|E| \geq \max_{i=h+1,\ldots,h'} |E_i| \tag{11}$$

Proof: To simplify our discussion, let $h = 0$ and $h' = n$. Now $T_{0,n} = T$. We assume that $T = T_1 \times T_2 \times \ldots T_k$, where T_i is an elementary trellis in KV construction for some $i = 1, \ldots, k$. $e_i^{out} = (e_{i0}^{out}, e_{i1}^{out}, \ldots, e_{i,n-1}^{out})$, is out-degree vector for elementary trellis T_i. It follows from from Eq. (4) that node's outdegree for time i in T is $q^{\sum_{j=1}^{k} e_{ji}^{out}}$. By corollary 1, we have that

$$\widetilde{T} = (\widetilde{T_1 \times T_2 \ldots \times T_k}) = \widetilde{T}_1 \times \widetilde{T}_2 \ldots \times \widetilde{T}_k. \tag{12}$$

The outdegree for $\widetilde{T}_t$ is $q^{\sum_{i=0}^{n-1} e_{ti}^{out}}$. It follows that outdegree of each node in $\widetilde{T}$ is

$q^{\sum_{t=1}^{k}\sum_{i=0}^{n-1} e_{ti}^{out}}$. Thus

$$|E| = |V_0| \times q^{\sum_{t=1}^{k}\sum_{i=0}^{n-1} e_{ti}^{out}}. \quad (13)$$

and for any m,

$$|E_m| = |V_{m-1}| \times q^{\sum_{j=1}^{k} e_{j,m-1}^{out}}. \quad (14)$$

Therefore

$$\begin{aligned}
\frac{|E|}{|E_{m|}} &= \frac{|V_0|}{|V_{m-1}|} \times q^{\sum_{t=1}^{k}\sum_{i=0,i\neq m-1}^{n-1} e_{ti}^{out}} \\
&= \prod_{l=1}^{m-1}\left(\frac{|V_{l-1}|}{|V_l|}\right) \times q^{\sum_{t=1}^{k}\sum_{i=0,i\neq m-1}^{n-1} e_{ti}^{out}} \\
&= \prod_{l=1}^{m-1}\left(\frac{|V_{l-1}| \times q^{\sum_{t=1}^{k} e_{t,l-1}^{out}}}{|V_l|}\right) \times q^{\sum_{t=1}^{k}\sum_{i=m}^{n-1} e_{ti}^{out}} \\
&= \prod_{l=1}^{m-1}\frac{|E_l|}{|V_l|} \times \prod_{i=m}^{n-1}\frac{|E_{i+1}|}{|V_i|}. \quad (15)
\end{aligned}$$

Since $\frac{|E_l|}{|V_l|}$ is the indegree of node in V_l and $\frac{|E_{i+1}|}{|V_i|}$ is the outdegree of node in V_i, Eq. (15) is greater or equal to 1. Note that our proof do not use the fact that $\sum_{i=0}^{n-1} e_{ti}^{out} = 1$ for each t or any such property concerned with the whole trellis. Therefore, it is valid for any h and h' by treating T_i as the corresponding section of the elementary trellis. In other word, it can be applied with slight change for any section of a linear tail-biting trellis. The only advantage of our assumption is to make notation less complicated. ❑

In the proof of Proposition 2, the equality holds if there exists s such that both $\frac{|E_l|}{|V_l|}$ = 1 for $l = h + 1, \ldots, s - 1$ and $\frac{|E_{l+1}|}{|V_l|}$ =1 for $l = s, \ldots, h' - 1$. Next theorem shows the necessary condition of an optimal sectionalization.

Theorem 1: Suppose that $T_{h,h'} = T_{h+1} * \ldots * T_{h'}$ is the section of linear tail-biting trellis T. Then $T_{h,h'}$ is not an optimal sectionalization if a strict inequality holds in Proposition 2.

***Proof*:** Suppose that $T_{h,h'}$ is an optimal sectionalization such that a strict inequality holds in Eq. (11), we will have a contradiction. Since for each i, strict inequality in Eq. (11) holds, we will have that either $\prod_{l=h+1}^{i-1} \frac{|E_l|}{|V_l|} > 1$ or $\prod_{j=i}^{h'-1} \frac{|E_{j+1}|}{|V_j|} > 1$. Without loss of generality, suppose that $\prod_{l=h+1}^{i-1} \frac{|E_l|}{|V_l|} > 1$ and m is the smallest integer such that $\prod_{l=h+1}^{m} \frac{|E_l|}{|V_l|} \geq q$. Now we get two subsections by decomposing T into

$$T' = T_{h,m} = T_{h+1} * T_{h+2} * \ldots * T_m$$

$$T'' = T_{m,h'} = T_{m+1} * T_{m+2} * \ldots * T_{h'} \tag{16}$$

Their total number of edges are

$$|E'| = |V_h| \times q^{\sum_{t=1}^{k} \sum_{i=h}^{m-1} e_{ti}^{out}}$$

$$|E''| = |V_m| \times q^{\sum_{t=1}^{k} \sum_{i=m}^{h'-1} e_{ti}^{out}} \tag{17}$$

It follows that

$$\begin{aligned}
\frac{|E|}{|E''|} &= \frac{|V_h|}{|V_m|} \times q^{\sum_{t=1}^{k} \sum_{i=h}^{m-1} e_{ti}^{out}} \\
&= \frac{|V_h|}{|V_m|} \times \prod_{i=h}^{m-1} \frac{|E_{i+1}|}{|V_i|} \\
&= \frac{|V_h|}{|V_m|} \times \prod_{i=h}^{m-1} \frac{|E_{i+1}|}{|V_{i+1}|} \times \prod_{i=h}^{m-1} \frac{|V_{i+1}|}{|V_i|} \\
&= \prod_{i=h}^{m-1} \frac{|E_{i+1}|}{|V_{i+1}|} \\
&= \prod_{i=h+1}^{m} \frac{|E_i|}{|V_i|} \geq q
\end{aligned} \tag{18}$$

If $\frac{|E|}{|E'|} = 1$, we get

$$\begin{aligned}
\frac{|E|}{|E_m|} &= \prod_{l=h+1}^{m-1} \frac{|E_l|}{|V_l|} \times \prod_{i=m}^{h'-1} \frac{|E_{i+1}|}{|V_i|} \\
&= \frac{|E|}{|E'|} = 1
\end{aligned} \tag{19}$$

Eq. (19) uses the fact that $\prod_{l=h+1}^{m-1} \frac{|E_l|}{|V_l|} = 1$ since m is the smallest integer such that $\prod_{l=h+1}^{i-1} \frac{|E_l|}{|V_l|} \geq q$ for $h + 1 \leq i \leq h'$. It contradicts our assumption of strict inequality (11) because of $|E| = |E_m|$. Therefore, $\frac{|E|}{|E'|} \geq q$. Thus, $|E| \geq |E'| + |E''|$ which contradicts the optimal sectionalization of $T_{h,h'}$ in T by Lemma 1. We can conclude that optimal sectionalization must hold the bound in Eq. (11) with equality. ❑

We now illustrate some examples of sectionalization in linear tail-biting trellis.

Example 1: Fig. 2 appears in [1] as an example of tail-biting trellis. We give two sectionalization of it which satisfies necessary condition of optimal sectionalization. Suppose that $T = T_1 \circ T_2 \circ \ldots \circ T_7$ in Fig. 2 where T_i is a section with length one. Then $T' = T_1 \circ T_2 \circ (T_3 * T_4) \circ T_5 \circ T_6 \circ T_7$ in Fig. 3 and $T'' = T_1 \circ (T_2 * T_3 * T_4) \circ (T_5 * T_6) \circ T_7$ in Fig. 4.

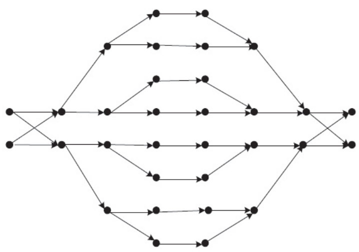

Fig. 2. Tail-biting trellis for a (7, 4, 2) binary linear code.

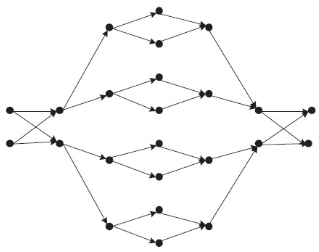

Fig. 3. $T' = T_1 \circ T_2 \circ (T_3 * T_4) \circ T_5 \circ T_6 \circ T_7$.

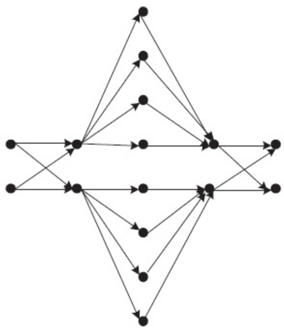

Fig. 4. $T'' = T_1 \circ (T_2 * T_3 * T_4) \circ (T_5 * T_6) \circ T_7$.

Although both trellises satisfy the necessary condition of optimal sectionalization, T'' is obviously much simpler and has less decoding complexity. Therefore, the necessary condition of optimal sectionalization can not be a sufficient condition. Moreover, we see that through sectionalization, T'' has a simpler structure than original trellis T. Sectionalization does improve the decoding efficiency in this case.

Example 2: In contrast, we present the sectionalization of conventional trellis for Example 1 where we assume $T = T_1 \circ T_2 \circ \ldots \circ T_7$ in Fig. 5. Fig. 5 appears in [1] compared with tali-biting trellis. Fig. 6 is optimal sectionalization of it. Although the state complexity for each time axis is not larger than tail-biting trellis in Fig. 2, its sectionalization is more complicated and of less efficiency compared with Fig. 4.

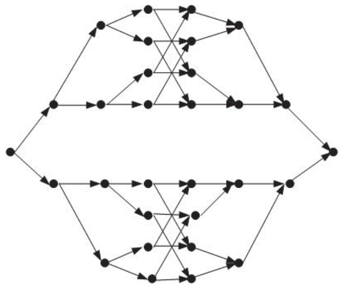

Fig. 5. Conventional trellis for a (7, 4, 2) binary linear code.

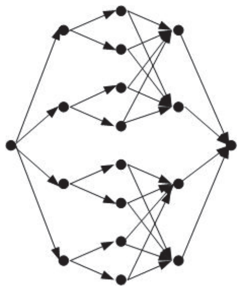

Fig. 6. $T''' = T_1 \circ (T_2 * T_3) \circ T_4 \circ (T_5 * T_6) \circ T_7$.

One of the reason is that we take edges into our consideration of decoding complexity. However, it is still unknown that which form of tail-biting trellis can achieve a smaller sectionalization. Since it is unrealistic to check all minimal tail-biting trellis, we need some efficient algorithm to deal with it.

4. CONCLUSIONS

In this paper, we investigate the property of tail-biting trellis. We prove that the sectionalization technique from conventional trellis is also hold for tail-biting trellis. Moreover, we give a necessary condition for optimal sectionalization of tail-biting trellis.

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